

# Genus-2 Gromov-Witten invariants for manifolds with semisimple quantum cohomology

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In [L2], the author studied universal equations for genus-2 Gromov-Witten invariants given in [Ge1] and [BP] using quantum product on the big phase space. Among other results, the author proved that for manifolds with semisimple quantum cohomology, the generating function for genus-2 Gromov-Witten invariants, denoted by  $F_2$ , is uniquely determined by known genus-2 universal equations. Moreover, an explicit formula for  $F_2$  was given in terms of genus-0 and genus-1 invariants. However, the formula given in [L2] is very complicated to work with. In this paper, we will give a much simpler formula using idempotents of the quantum product on the big phase space, and then use it to prove the genus-2 Virasoro conjecture for manifolds with semisimple quantum cohomology (cf. [EHX] and [CK]).

Properties of idempotents on the big phase space were studied in [L4]. Let  $M$  be a compact symplectic manifold. In Gromov-Witten theory, the space  $H^*(M; \mathbb{C})$  is called the *small phase space*. A product of infinitely many copies of the small phase space is called the *big phase space*. The generating functions for Gromov-Witten invariants are formal power series on the big phase space. Let  $N$  be the dimension of  $H^*(M; \mathbb{C})$ . If the quantum cohomology of  $M$  is semisimple, there exist vector fields  $\mathcal{E}_i$ ,  $i = 1, \dots, N$ , on the big phase space such that  $\mathcal{E}_i \circ \mathcal{E}_j = \delta_{ij} \mathcal{E}_i$  for all  $i$  and  $j$ , where “ $\circ$ ” stands for the quantum product (see equation (1)). These vector fields are called *idempotents*. Let  $u_i$ ,  $i = 1, \dots, N$ , be the eigenvalues of the quantum multiplication by the Euler vector field (see equation (2)). The first main result of this paper is the following

**Theorem 0.1** *If the quantum cohomology of the underlying manifold is semisimple, the genus-2 generating function  $F_2$  is given by*

$$F_2 = \frac{1}{2}A_1(\tau_-(\mathcal{S})) + \frac{1}{3}A_1(\tau_-^2(\mathcal{L}_0)) - \frac{1}{6} \sum_{i=1}^N u_i B(\mathcal{E}_i, \mathcal{E}_i, \mathcal{E}_i).$$

In this formula,  $A_1$  and  $B$  are tensors which only depend on genus-0 and genus-1 data. Precise formulas for  $A_1$  and  $B(\mathcal{E}_i, \mathcal{E}_i, \mathcal{E}_i)$  are given in equation (15) and equation (27) respectively. The tensor  $A_1$  comes from the genus-0 and genus-1 part of the genus-2 topological recursion relation derived from Mumford’s relation (cf. [Ge1]) and tensor  $B$  comes

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from the genus-0 and genus-1 part of an equation due to Belorousski and Pandharipande (cf. [BP]).  $\mathcal{S}$  is the string vector field (see equation (3)), and  $\mathcal{L}_n$  is the  $n$ -th Virasoro vector field (see equation (17)). The operator  $\tau_-$  lower the level of descendants of vector fields by 1. We will also give a formula for  $F_2$  which only involves genus-0 data in Theorem 3.1.

The Virasoro conjecture predicts that the generating functions of Gromov-Witten invariants of smooth projective varieties are annihilated by an infinite sequence of differential operators which form a half branch of the Virasoro algebra. This conjecture was proposed by Eguchi-Hori-Xiong and Katz (cf. [EHX], [CK]). It is a natural generalization of Witten's KdV conjecture (cf. [W1] [W2]) which was proved by Kontsevich (cf. [Ko]). For manifolds with semisimple quantum cohomology, the Virasoro conjecture completely determines the higher genus Gromov-Witten invariants in terms of genus-0 invariants (cf. [DZ3]). In [LT], Tian and the author proved the genus-0 Virasoro conjecture for all compact symplectic manifolds (see also [DZ2], [Ge2], [L3], [Gi3]). The genus-1 Virasoro conjecture for manifolds with semisimple quantum cohomology was proved by Dubrovin and Zhang [DZ2] (see also [L1] and [L4]). In [L1] and [L2], the author also proved that the genus-1 and genus-2 Virasoro conjecture for all smooth projective varieties can be reduced to an  $SL(2)$  symmetry of Gromov-Witten invariants. The second main result of this paper is the following

**Theorem 0.2** *For smooth projective varieties with semisimple quantum cohomology, the genus-2 Virasoro conjecture is true.*

In [Gi1], Givental conjectured a formula for higher genus Gromov-Witten potential for manifolds with semisimple quantum cohomology. His formula satisfies the Virasoro constraints (cf. [Gi2]). Since in the semisimple case, Virasoro constraints uniquely determine the higher genus Gromov-Witten potential (cf. [DZ3]), Theorem 0.2 implies that Givental's conjectural formula is correct in the genus-2 case. We also note that the method used in this paper should apply to higher genus case once the corresponding universal equations are obtained.

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## 1 Quantum product and idempotents

We first review properties of quantum product and idempotents on the big phase space which will be used in this paper. The proofs for these properties can be found in [L2] and [L4].

Let  $M$  be a compact symplectic manifold. For simplicity, we assume  $H^{\text{odd}}(M; \mathbb{C}) = 0$ . The *big phase space* is by definition the product of infinite copies of  $H^*(M; \mathbb{C})$ , i.e.

$$P := \prod_{n=0}^{\infty} H^*(M; \mathbb{C}).$$

Fix a basis  $\{\gamma_1, \dots, \gamma_N\}$  of  $H^*(M; \mathbb{C})$  with  $\gamma_1 = 1$  being the identity of the ordinary cohomology ring of  $M$ . Then we denote the corresponding basis for the  $n$ -th copy of  $H^*(M; \mathbb{C})$  in  $P$  by  $\{\tau_n(\gamma_1), \dots, \tau_n(\gamma_N)\}$ . We call  $\tau_n(\gamma_\alpha)$  a *descendant* of  $\gamma_\alpha$  with descendant level  $n$ . We can think of  $P$  as an infinite dimensional vector space with basis  $\{\tau_n(\gamma_\alpha) \mid 1 \leq \alpha \leq N, n \in \mathbb{Z}_{\geq 0}\}$  where  $\mathbb{Z}_{\geq 0} = \{n \in \mathbb{Z} \mid n \geq 0\}$ . Let  $(t_n^\alpha \mid 1 \leq \alpha \leq N, n \in \mathbb{Z}_{\geq 0})$  be the corresponding coordinate system on  $P$ . For convenience, we identify  $\tau_n(\gamma_\alpha)$  with the coordinate vector field  $\frac{\partial}{\partial t_n^\alpha}$  on  $P$  for  $n \geq 0$ . If  $n < 0$ ,  $\tau_n(\gamma_\alpha)$  is understood as the 0 vector field. We also abbreviate  $\tau_0(\gamma_\alpha)$  as  $\gamma_\alpha$ . Any vector field of the form  $\sum_\alpha f_\alpha \gamma_\alpha$ , where  $f_\alpha$  are functions on the big phase space, is called a *primary vector field*. We use  $\tau_+$  and  $\tau_-$  to denote the operator which shift the level of descendants, i.e.

$$\tau_\pm \left( \sum_{n,\alpha} f_{n,\alpha} \tau_n(\gamma_\alpha) \right) = \sum_{n,\alpha} f_{n,\alpha} \tau_{n\pm 1}(\gamma_\alpha)$$

where  $f_{n,\alpha}$  are functions on the big phase space.

We will use the following *conventions for notations*: All summations are over the entire meaningful ranges of the indices unless otherwise indicated. Let

$$\eta_{\alpha\beta} = \int_M \gamma_\alpha \cup \gamma_\beta$$

be the intersection form on  $H^*(M, \mathbb{C})$ . We will use  $\eta = (\eta_{\alpha\beta})$  and  $\eta^{-1} = (\eta^{\alpha\beta})$  to lower and raise indices. For example,

$$\gamma^\alpha := \eta^{\alpha\beta} \gamma_\beta.$$

Here we are using the summation convention that repeated indices (in this formula,  $\beta$ ) should be summed over their entire ranges.

Let

$$\langle \tau_{n_1}(\gamma_{\alpha_1}) \tau_{n_2}(\gamma_{\alpha_2}) \dots \tau_{n_k}(\gamma_{\alpha_k}) \rangle_g$$

be the genus- $g$  descendant Gromov-Witten invariant associated to  $\gamma_{\alpha_1}, \dots, \gamma_{\alpha_k}$  and non-negative integers  $n_1, \dots, n_k$  (cf. [W1], [RT], [LiT]). The genus- $g$  generating function is defined to be

$$F_g = \sum_{k \geq 0} \frac{1}{k!} \sum_{\substack{\alpha_1, \dots, \alpha_k \\ n_1, \dots, n_k}} t_{n_1}^{\alpha_1} \dots t_{n_k}^{\alpha_k} \langle \tau_{n_1}(\gamma_{\alpha_1}) \tau_{n_2}(\gamma_{\alpha_2}) \dots \tau_{n_k}(\gamma_{\alpha_k}) \rangle_g.$$

This function is understood as a formal power series of  $t_n^\alpha$ .

Introduce a  $k$ -tensor  $\langle \underbrace{\dots}_k \rangle$  defined by

$$\langle \mathcal{W}_1 \mathcal{W}_2 \dots \mathcal{W}_k \rangle_g := \sum_{m_1, \alpha_1, \dots, m_k, \alpha_k} f_{m_1, \alpha_1}^1 \dots f_{m_k, \alpha_k}^k \frac{\partial^k}{\partial t_{m_1}^{\alpha_1} \partial t_{m_2}^{\alpha_2} \dots \partial t_{m_k}^{\alpha_k}} F_g,$$

for vector fields  $\mathcal{W}_i = \sum_{m,\alpha} f_{m,\alpha}^i \frac{\partial}{\partial t_m^\alpha}$  where  $f_{m,\alpha}^i$  are functions on the big phase space. We can also view this tensor as the  $k$ -th covariant derivative of  $F_g$  with respect to the trivial

connection on  $P$ . This tensor is called the  $k$ -point (*correlation*) *function*. For any vector fields  $\mathcal{W}_1$  and  $\mathcal{W}_2$  on the big phase space, the *quantum product* of  $\mathcal{W}_1$  and  $\mathcal{W}_2$  is defined by

$$\mathcal{W}_1 \circ \mathcal{W}_2 := \langle\langle \mathcal{W}_1 \mathcal{W}_2 \gamma^\alpha \rangle\rangle_0 \gamma_\alpha. \quad (1)$$

This is a commutative and associative product. But it does not have an identity. For any vector field  $\mathcal{W}$  and integer  $k \geq 1$ ,  $\mathcal{W}^k$  is understood as the  $k$ -th power of  $\mathcal{W}$  with respect to this product.

Let

$$\mathcal{X} := - \sum_{m,\alpha} (m + b_\alpha - b_1 - 1) \tilde{t}_m^\alpha \tau_m(\gamma_\alpha) - \sum_{m,\alpha,\beta} \mathcal{C}_\alpha^\beta \tilde{t}_m^\alpha \tau_{m-1}(\gamma_\beta) \quad (2)$$

be the Euler vector field on the big phase space  $P$ , where  $\tilde{t}_m^\alpha = t_m^\alpha - \delta_{m,1} \delta_{\alpha,1}$ ,

$$b_\alpha = \frac{1}{2}(\text{dimension of } \gamma_\alpha) - \frac{1}{4}(\text{real dimension of } M) + \frac{1}{2}$$

and the matrix  $\mathcal{C} = (\mathcal{C}_\alpha^\beta)$  is defined by  $c_1(V) \cup \gamma_\alpha = \mathcal{C}_\alpha^\beta \gamma_\beta$ . For smooth projective varieties, the dimension of  $\gamma_\alpha$  should be replaced by twice of the holomorphic dimension of  $\gamma_\alpha$  in the definition of  $b_\alpha$ .

The quantum multiplication by  $\mathcal{X}$  is an endomorphism on the space of primary vector fields on  $P$ . If this endomorphism has distinct eigenvalues at generic points, we call  $P$  *semisimple*. In this case, let  $\mathcal{E}_1, \dots, \mathcal{E}_N$  be the eigenvectors with corresponding eigenvalues  $u_1, \dots, u_N$ , i.e.

$$\mathcal{X} \circ \mathcal{E}_i = u_i \mathcal{E}_i$$

for each  $i = 1, \dots, N$ .  $\mathcal{E}_i$  is considered as a vector field on  $P$ , and  $u_i$  is considered as a function on  $P$ . They satisfy the following properties:

$$\mathcal{E}_i \circ \mathcal{E}_j = \delta_{ij} \mathcal{E}_i, \quad [\mathcal{E}_i, \mathcal{E}_j] = 0, \quad \mathcal{E}_i u_j = \delta_{ij}$$

for any  $i$  and  $j$ . We call  $\{\mathcal{E}_1, \dots, \mathcal{E}_N\}$  *idempotents* on the big phase space. When restricted to the small phase space, they coincide with the coordinate vector fields of the canonical coordinate system of semisimple Frobenius manifolds (cf. [D]).

Let

$$\mathcal{S} := - \sum_{m,\alpha} \tilde{t}_m^\alpha \tau_{m-1}(\gamma_\alpha) \quad (3)$$

be the *string vector field* on  $P$ . We define

$$\overline{\mathcal{W}} = \mathcal{W} \circ \mathcal{S}$$

for any vector field  $\mathcal{W}$  on  $P$ . The vector field  $\overline{\mathcal{S}}$  is the identity for the quantum product when restricted to the space of primary vector fields. We have

$$\overline{\mathcal{S}} = \sum_{i=1}^N \mathcal{E}_i. \quad (4)$$

and

$$\overline{\mathcal{X}}^k = \sum_{i=1}^N u_i^k \mathcal{E}_i. \quad (5)$$

for  $k \geq 1$ .

For any vector fields  $\mathcal{W}$  and  $\mathcal{V}$  on the big phase space, define

$$\langle \mathcal{W}, \mathcal{V} \rangle := \langle \langle \mathcal{S} \mathcal{W} \mathcal{V} \rangle \rangle_0. \quad (6)$$

This bilinear form generalizes the Poincare metric on the small phase space. It is nondegenerate only when restricted to the space of primary vector fields. It is also compatible with quantum product in the following sense:

$$\langle (\mathcal{W}_1 \circ \mathcal{W}_2), \mathcal{W}_3 \rangle = \langle \mathcal{W}_2, (\mathcal{W}_1 \circ \mathcal{W}_3) \rangle, \quad (7)$$

and

$$\langle \mathcal{W}_1, \mathcal{W}_2 \rangle = \langle \overline{\mathcal{W}}_1, \overline{\mathcal{W}}_2 \rangle$$

for any vector fields  $\mathcal{W}_i$ . We have  $\langle \mathcal{E}_i, \mathcal{E}_j \rangle = 0$  if  $i \neq j$  and the functions

$$g_i := \langle \mathcal{E}_i, \mathcal{E}_i \rangle$$

are non-zero in the region where idempotents are well defined. Any primary vector field  $\mathcal{W}$  has the decomposition

$$\mathcal{W} = \sum_{i=1}^N \frac{\langle \mathcal{W}, \mathcal{E}_i \rangle}{g_i} \mathcal{E}_i.$$

Let  $\nabla$  be the covariant derivative on  $P$  of the trivial flat connection with respect to the standard coordinates  $\{t_n^\alpha\}$ . The compatibilities of this connection with the quantum product and the bilinear form are given by

$$\nabla_{\mathcal{W}_1}(\mathcal{W}_2 \circ \mathcal{W}_3) = (\nabla_{\mathcal{W}_1} \mathcal{W}_2) \circ \mathcal{W}_3 + \mathcal{W}_2 \circ (\nabla_{\mathcal{W}_1} \mathcal{W}_3) + \langle \langle \mathcal{W}_1 \mathcal{W}_2 \mathcal{W}_3 \gamma^\alpha \rangle \rangle_0 \gamma_\alpha \quad (8)$$

and

$$\begin{aligned} \mathcal{W}_1 \langle \mathcal{W}_2, \mathcal{W}_3 \rangle &= \langle \{\nabla_{\mathcal{W}_1} \mathcal{W}_2 + \mathcal{W}_1 \circ \tau_-(\mathcal{W}_2)\}, \mathcal{W}_3 \rangle \\ &+ \langle \mathcal{W}_2, \{\nabla_{\mathcal{W}_1} \mathcal{W}_3 + \mathcal{W}_1 \circ \tau_-(\mathcal{W}_3)\} \rangle \end{aligned} \quad (9)$$

for any vector fields  $\mathcal{W}_i$ . Equation (9) suggests that the modified connection  $\tilde{\nabla}$  defined by

$$\tilde{\nabla}_{\mathcal{W}_1} \mathcal{W}_2 := \nabla_{\mathcal{W}_1} \mathcal{W}_2 + \mathcal{W}_1 \circ \tau_-(\mathcal{W}_2)$$

is compatible with the bilinear form  $\langle \cdot, \cdot \rangle$ . Moreover equation (8) implies that the family of connections  $\tilde{\nabla}^z$  defined by

$$\tilde{\nabla}_{\mathcal{W}_1}^z \mathcal{W}_2 := \nabla_{\mathcal{W}_1} \mathcal{W}_2 + z \mathcal{W}_1 \circ \tau_-(\mathcal{W}_2)$$

are flat for all  $z$ , where  $z$  is an arbitrary parameter.

Covariant derivatives of idempotents are given by

$$\nabla_{\mathcal{W}} \mathcal{E}_i = -2 \langle\langle \mathcal{W} \mathcal{E}_i \mathcal{E}_i \gamma^\alpha \rangle\rangle_0 \gamma_\alpha \circ \mathcal{E}_i + \langle\langle \mathcal{W} \mathcal{E}_i \mathcal{E}_i \gamma^\alpha \rangle\rangle_0 \gamma_\alpha$$

for any vector field  $\mathcal{W}$ . In particular,

$$\nabla_{\mathcal{E}_j} \mathcal{E}_i = \delta_{ij} \mathcal{F}_j - \mathcal{F}_j \circ \mathcal{E}_i - \mathcal{F}_i \circ \mathcal{E}_j$$

where

$$\mathcal{F}_j := \langle\langle \mathcal{E}_j \mathcal{E}_j \mathcal{E}_j \gamma^\alpha \rangle\rangle_0 \gamma_\alpha$$

for each  $j = 1, \dots, N$ . Vector fields  $\mathcal{F}_j$  are also related to the string vector field by the formula

$$\overline{\tau_-(\mathcal{S})} = - \sum_{i=1}^N \mathcal{F}_i. \quad (10)$$

For any vector field  $\mathcal{W}$ , define

$$T(\mathcal{W}) := \tau_+(\mathcal{W}) - \mathcal{S} \circ \tau_+(\mathcal{W}).$$

The operator  $T$  was introduced in [L2] to simplify topological recursion relations for Gromov-Witten invariants. It corresponds to the  $\psi$  classes in the relations in the tautological ring of moduli space of stable curves. In some sense, repeatedly applying  $T$  to a vector field will trivialize its action on genus- $g$  generating functions. Here are some basic properties of  $T$ : For any vector fields  $\mathcal{W}_i$ ,

- (i)  $T(\mathcal{W}_1) \circ \mathcal{W}_2 = 0$ ,
- (ii)  $\langle\langle T(\mathcal{W}_1) \mathcal{W}_2 \mathcal{W}_3 \mathcal{W}_4 \rangle\rangle_0 = \langle\langle (\mathcal{W}_1 \circ \mathcal{W}_2) \mathcal{W}_3 \mathcal{W}_4 \rangle\rangle_0$
- (iii)  $\nabla_{\mathcal{W}_1} T(\mathcal{W}_2) = T(\nabla_{\mathcal{W}_1} \mathcal{W}_2) - \mathcal{W}_1 \circ \mathcal{W}_2$
- (iv)  $T(\mathcal{W}) u_i = 0$ ,
- (v)  $\langle T(\mathcal{W}_1), \mathcal{W}_2 \rangle = 0$ ,
- (vi)  $\nabla_{T(\mathcal{W})} \mathcal{E}_i = -\mathcal{W} \circ \mathcal{E}_i$ ,
- (vii)  $[T(\mathcal{W}), \mathcal{E}_i] = -T(\nabla_{\mathcal{E}_i} \mathcal{W})$ .

Note that  $\{T^k(\mathcal{E}_i) \mid i = 1, \dots, N, k \geq 0\}$  gives a frame for the tangent bundle of the big phase space. This frame is not commutative due to property (vii). Any vector field  $\mathcal{W}$  has the following decomposition

$$\mathcal{W} = T^k(\tau_-^k(\mathcal{W})) + \sum_{i=0}^{k-1} T^i(\overline{\tau_-^i(\mathcal{W})}) \quad (11)$$

where  $k$  is any positive integer (cf [L2, Equation (26)]). This decomposition is very useful when applying topological recursion relations. In particular, we will frequently use the decomposition

$$\mathcal{W} = \overline{\mathcal{W}} + T(\tau_-(\mathcal{W}))$$

and call this the *standard decomposition* of  $\mathcal{W}$ . For example, using this decomposition, we see

$$\mathcal{W} u_i = \overline{\mathcal{W}} u_i$$

for any vector field  $\mathcal{W}$  on the big phase space.

For any vector field  $\mathcal{W} = \sum_{n,\alpha} f_{n,\alpha} \tau_n(\gamma_\alpha)$ , define

$$\mathcal{G} * \mathcal{W} := \sum_{n,\alpha} (n + b_\alpha) f_{n,\alpha} \tau_n(\gamma_\alpha).$$

This operator was used in [L2] to give a recursive description for the Virasoro vector fields. On the space of primary vector fields, the operator  $\mathcal{G}*$  has the following property:

$$\langle \mathcal{G} * \mathcal{W}, \mathcal{V} \rangle + \langle \mathcal{W}, \mathcal{G} * \mathcal{V} \rangle = \langle \mathcal{W}, \mathcal{V} \rangle \quad (12)$$

for all primary vector fields  $\mathcal{V}$  and  $\mathcal{W}$ . Moreover, for any  $i$ ,

$$\mathcal{G} * \mathcal{E}_i = \frac{1}{2} \mathcal{E}_i - u_i \mathcal{F}_i + \mathcal{X} \circ \mathcal{F}_i. \quad (13)$$

## 2 Universal equations in genus 2

We will need two genus-2 universal equations. The first one is the genus-2 topological recursion relation derived from Mumford's relation (cf. [Ge1]): For any vector field  $\mathcal{W}$ ,

$$\langle\langle T^2(\mathcal{W}) \rangle\rangle_2 = A_1(\mathcal{W}) \quad (14)$$

where

$$\begin{aligned} A_1(\mathcal{W}) &:= \frac{7}{10} \langle\langle \gamma_\alpha \rangle\rangle_1 \langle\langle \{\gamma^\alpha \circ \mathcal{W}\} \rangle\rangle_1 + \frac{1}{10} \langle\langle \gamma_\alpha \{\gamma^\alpha \circ \mathcal{W}\} \rangle\rangle_1 - \frac{1}{240} \langle\langle \mathcal{W} \{\gamma_\alpha \circ \gamma^\alpha\} \rangle\rangle_1 \\ &\quad + \frac{13}{240} \langle\langle \mathcal{W} \gamma_\alpha \gamma^\alpha \gamma^\beta \rangle\rangle_0 \langle\langle \gamma_\beta \rangle\rangle_1 + \frac{1}{960} \langle\langle \mathcal{W} \gamma^\alpha \gamma_\alpha \gamma^\beta \gamma_\beta \rangle\rangle_0. \end{aligned} \quad (15)$$

Another genus-2 equation is the following (cf. [BP]): For any vector fields  $\mathcal{W}_i$ ,

$$\begin{aligned} &2 \langle\langle \{\mathcal{W}_1 \circ \mathcal{W}_2 \circ \mathcal{W}_3\} \rangle\rangle_2 - 2 \langle\langle \mathcal{W}_1 \mathcal{W}_2 \mathcal{W}_3 \gamma^\alpha \rangle\rangle_0 \langle\langle T(\gamma_\alpha) \rangle\rangle_2 \\ &+ \frac{1}{2} \sum_{\sigma \in S_3} \left\{ \langle\langle \mathcal{W}_{\sigma(1)} T(\mathcal{W}_{\sigma(2)} \circ \mathcal{W}_{\sigma(3)}) \rangle\rangle_2 - \langle\langle T(\mathcal{W}_{\sigma(1)}) \{\mathcal{W}_{\sigma(2)} \circ \mathcal{W}_{\sigma(3)}\} \rangle\rangle_2 \right\} \\ &= B(\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3), \end{aligned} \quad (16)$$

where  $B$  is a symmetric 3-tensor which only depends on genus 0 and genus 1 data. The precise definition for  $B$  is very complicated (see [L2, Section 2]). In this paper, we only need the special case  $B(\mathcal{E}_i, \mathcal{E}_i, \mathcal{E}_i)$ . We will give the definition for this function in equation (27).

We will also need the Virasoro vector fields  $\mathcal{L}_n$ . A recursive description for these vector fields were given in [L2] by using an operator  $R$ . For any vector field  $\mathcal{W} = \sum_{n,\alpha} f_{n,\alpha} \tau_n(\gamma_\alpha)$ ,

define  $C(W) := \sum_{n,\alpha,\beta} f_{n,\alpha} \mathcal{C}_\alpha^\beta \tau_n(\gamma_\beta)$ , where  $\mathcal{C}$  is the matrix of multiplication by the first Chern class  $c_1(M)$  in the ordinary cohomology ring. Define

$$R(\mathcal{W}) := \mathcal{G} * T(\mathcal{W}) + C(\mathcal{W}).$$

Then the Virasoro vector fields are given by

$$\mathcal{L}_n := -R^{n+1}(\mathcal{S}) \quad (17)$$

for  $n \geq -1$ . One of the nice properties of  $\mathcal{L}_n$  is

$$\overline{\mathcal{L}}_n = -\overline{\mathcal{X}}^{n+1}$$

for  $n \geq -1$  (cf. [L2, Lemma 4.1]). Here,  $\overline{\mathcal{X}}^0$  is understood as  $\overline{\mathcal{S}}$ .

**Theorem 2.1** *For  $k \geq -1$ ,*

$$\overline{\tau_-(\mathcal{L}_k)} = \sum_{i=1}^N u_i^{k+1} \mathcal{F}_i - \frac{3}{2}(k+1) \overline{\mathcal{X}}^k.$$

**Proof:** We prove this theorem by induction on  $k$ . First note that for  $k = -1$ , this theorem is precisely equation (10). Secondly, by equation (13), we have

$$\mathcal{G} * \overline{\mathcal{X}}^k = \sum_{i=1}^N u_i^k \mathcal{G} * \mathcal{E}_i = \frac{1}{2} \overline{\mathcal{X}}^k - \sum_{i=1}^N u_i^{k+1} \mathcal{F}_i + \overline{\mathcal{X}} \circ \sum_{i=1}^N u_i^k \mathcal{F}_i.$$

By [L2, Lemma 4.1 and Theorem 4.8],

$$\overline{\tau_-(\mathcal{L}_k)} = \overline{\mathcal{X}} \circ \overline{\tau_-(\mathcal{L}_{k-1})} - \overline{\mathcal{X}}^k - \mathcal{G} * \overline{\mathcal{X}}^k.$$

Applying the induction hypothesis, we obtain the desired formula.  $\square$

**Corollary 2.2** *For  $k \geq -1$ ,*

$$\mathcal{L}_k = \sum_{i=1}^N u_i^{k+1} (T(\mathcal{F}_i) - \mathcal{E}_i) - \frac{3}{2}(k+1) T(\overline{\mathcal{X}}^k) + T^2(\tau_-^2(\mathcal{L}_k)).$$

**Proof:** This follows from the following special case of equation (11):

$$\mathcal{L}_k = \overline{\mathcal{L}}_k + T(\overline{\tau_-(\mathcal{L}_k)}) + T^2(\tau_-^2(\mathcal{L}_k)) \quad (18)$$

and the fact

$$\overline{\mathcal{L}}_k = -\overline{\mathcal{X}}^{k+1} = -\sum_{i=1}^N u_i^{k+1} \mathcal{E}_i.$$

$\square$

**Lemma 2.3**

$$\langle\langle T(\overline{\mathcal{S}}) \rangle\rangle_2 = \frac{2}{3}A_1(\tau_-^2(\mathcal{L}_0)) - \frac{1}{3}\sum_{i=1}^N u_i B(\mathcal{E}_i, \mathcal{E}_i, \mathcal{E}_i).$$

**Proof:** In case  $\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_3 = \mathcal{E}_i$ , equation (16) has a much simpler form

$$\langle\langle \mathcal{E}_i \rangle\rangle_2 - \langle\langle T(\mathcal{F}_i) \rangle\rangle_2 = \frac{1}{2}B(\mathcal{E}_i, \mathcal{E}_i, \mathcal{E}_i). \quad (19)$$

On the other hand, the genus-2  $L_0$ -constraint has the form  $\langle\langle \mathcal{L}_0 \rangle\rangle_2 = 0$ . Corollary 2.2 and equation (14) then imply the following

$$0 = \langle\langle \mathcal{L}_0 \rangle\rangle_2 = \sum_{i=1}^N u_i (\langle\langle T(\mathcal{F}_i) \rangle\rangle_2 - \langle\langle \mathcal{E}_i \rangle\rangle_2) - \frac{3}{2} \langle\langle T(\overline{\mathcal{S}}) \rangle\rangle_2 + A_1(\tau_-^2(\mathcal{L}_0)).$$

The lemma then follows from equation (19).  $\square$

**Proof of Theorem 0.1:** Recall that the *dilaton vector field* has the form  $\mathcal{D} = T(\mathcal{S})$  (cf. the remark after [L2, lemma 1.4]). The standard decomposition of  $\mathcal{S}$  then gives

$$\mathcal{D} = T(\overline{\mathcal{S}} + T(\tau_-(\mathcal{S}))) = T(\overline{\mathcal{S}}) + T^2(\tau_-(\mathcal{S})).$$

By the genus-2 dilaton equation

$$2F_2 = \langle\langle \mathcal{D} \rangle\rangle_2 = \langle\langle T(\overline{\mathcal{S}}) \rangle\rangle_2 + \langle\langle T^2(\tau_-(\mathcal{S})) \rangle\rangle_2.$$

The theorem then follows from Lemma 2.3 and equation (14).  $\square$

**Remark:** A formula for  $F_2$  under a somewhat weaker condition was given in [L2, Theorem 5.17]. The formula given here is much simpler and much easier to work with than the corresponding formula in [L2]. Moreover, the proof given here is also much simpler than the proof in [L2].

**Theorem 2.4** *If the quantum cohomology is semisimple, then for  $k \geq -1$ ,*

$$\begin{aligned} \langle\langle \mathcal{L}_k \rangle\rangle_2 &= -\frac{1}{2}\sum_{i=1}^N u_i^{k+1} B(\mathcal{E}_i, \mathcal{E}_i, \mathcal{E}_i) + \frac{k+1}{4} T(\overline{\mathcal{X}}^k) \left\{ \sum_{i=1}^N u_i B(\mathcal{E}_i, \mathcal{E}_i, \mathcal{E}_i) \right\} \\ &\quad + A_1(\tau_-^2(\mathcal{L}_k)) - \frac{k+1}{2} T(\overline{\mathcal{X}}^k) A_1(\tau_-^2(\mathcal{L}_0)) - \frac{3(k+1)}{4} T(\overline{\mathcal{X}}^k) A_1(\tau_-(\mathcal{S})) \end{aligned}$$

**Proof:** By Corollary 2.2,

$$\langle\langle \mathcal{L}_k \rangle\rangle_2 = -\sum_{i=1}^N u_i^{k+1} (\langle\langle \mathcal{E}_i \rangle\rangle_2 - \langle\langle T(\mathcal{F}_i) \rangle\rangle_2) - \frac{3(k+1)}{2} T(\overline{\mathcal{X}}^k) F_2 + \langle\langle T^2(\tau_-^2(\mathcal{L}_k)) \rangle\rangle_2.$$

The theorem then follows from applying equation (19) to the first term, Theorem 0.1 to the second term, and equation (14) to the third term.  $\square$

So far we have used only special cases of equation (16). In fact we can not get more information on the genus-2 generating function from other cases of this equation. But we can still get some interesting properties of the complicated tensor  $B$  from studying the more general cases of this equation. We have the following

**Lemma 2.5** For  $i \neq j \neq k$ ,

- (a)  $B(\mathcal{E}_i, \mathcal{E}_j, \mathcal{E}_k) = 0$
- (b)  $B(\mathcal{E}_i, \mathcal{E}_j, \mathcal{E}_j) = -B(\mathcal{E}_i, \mathcal{E}_i, \mathcal{E}_j)$
- (c)  $B(\mathcal{E}_i, \mathcal{E}_i, \mathcal{E}_j) = \frac{1}{2}T(\mathcal{E}_i)B(\mathcal{E}_j, \mathcal{E}_j, \mathcal{E}_j) - \frac{1}{2}T(\mathcal{E}_j)B(\mathcal{E}_i, \mathcal{E}_i, \mathcal{E}_i) + A_2(\mathcal{F}_j, \mathcal{E}_i) - A_2(\mathcal{F}_i, \mathcal{E}_j)$

In this lemma,  $A_2$  is a symmetric 2-tensor which only depends on genus-0 and genus-1 data. It comes from a genus-2 equation due to Getzler which takes the following form (cf. [Ge1]): For any vector fields  $\mathcal{W}_i$ ,

$$\langle\langle T(\mathcal{W}_1) T(\mathcal{W}_2) \rangle\rangle_2 = A_2(\mathcal{W}_1, \mathcal{W}_2). \quad (20)$$

It was proved in [L2] that this equation follows from equation (14) and equation (16).

**Proof of Lemma 2.5:** It was proved in [L4] that the genus-0 4-point functions satisfy the following properties: For any  $i \neq j \neq k$  and any vector field  $\mathcal{W}$ ,

$$\begin{aligned} (i) \quad & \langle\langle \mathcal{E}_i \mathcal{E}_j \mathcal{E}_k \mathcal{W} \rangle\rangle_0 = 0, \\ (ii) \quad & \langle\langle \mathcal{W} \mathcal{E}_i \mathcal{E}_i \mathcal{E}_j \rangle\rangle_0 = -\langle\langle \mathcal{W} \mathcal{E}_j \mathcal{E}_j \mathcal{E}_i \rangle\rangle_0, \\ (iii) \quad & \langle\langle \mathcal{E}_i \mathcal{E}_i \mathcal{E}_j \gamma^\alpha \rangle\rangle_0 \gamma_\alpha = \mathcal{F}_j \circ \mathcal{E}_i - \mathcal{F}_i \circ \mathcal{E}_j. \end{aligned} \quad (21)$$

Applying equation (16) for  $\mathcal{W}_1 = \mathcal{E}_i$ ,  $\mathcal{W}_2 = \mathcal{E}_j$ ,  $\mathcal{W}_2 = \mathcal{E}_k$ , we have

$$B(\mathcal{E}_i, \mathcal{E}_j, \mathcal{E}_k) = -2 \langle\langle \mathcal{E}_i \mathcal{E}_j \mathcal{E}_k \gamma^\alpha \rangle\rangle_0 \langle\langle T(\gamma_\alpha) \rangle\rangle_2.$$

Therefore (a) follows from equation (21) (i).

Applying equation (16) for  $\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{E}_i$ ,  $\mathcal{W}_2 = \mathcal{E}_j$ , we have

$$B(\mathcal{E}_i, \mathcal{E}_i, \mathcal{E}_j) = -2 \langle\langle \mathcal{E}_i \mathcal{E}_i \mathcal{E}_j \gamma^\alpha \rangle\rangle_0 \langle\langle T(\gamma_\alpha) \rangle\rangle_2 - \langle\langle \mathcal{E}_i T(\mathcal{E}_j) \rangle\rangle_2 + \langle\langle \mathcal{E}_j T(\mathcal{E}_i) \rangle\rangle_2. \quad (22)$$

By equation (21) (ii), if we interchange  $i$  and  $j$ , the right hand side of this equation is only changed by a minus sign. This proves (b).

Since  $\nabla_{T(\mathcal{E}_j)} \mathcal{E}_i = 0$  for  $i \neq j$ ,

$$\nabla_{T(\mathcal{E}_j)} T(\mathcal{F}_i) = T(\nabla_{T(\mathcal{E}_j)} \mathcal{F}_i) = T(\langle\langle T(\mathcal{E}_j) \mathcal{E}_i \mathcal{E}_i \mathcal{E}_i \gamma^\alpha \rangle\rangle_0 \gamma_\alpha).$$

By [L2, Equation (9)], we have

$$\nabla_{T(\mathcal{E}_j)} T(\mathcal{F}_i) = T(\langle\langle \mathcal{E}_j \mathcal{E}_i \mathcal{E}_i \gamma^\alpha \rangle\rangle_0 \gamma_\alpha \circ \mathcal{E}_i) = T(\mathcal{F}_j \circ \mathcal{E}_i)$$

where the last equality follows from equation (21) (iii). Hence taking derivative of equation (19) along  $T(\mathcal{E}_j)$ , we obtain

$$\langle\langle \mathcal{E}_i T(\mathcal{E}_j) \rangle\rangle_2 - \langle\langle T(\mathcal{F}_j \circ \mathcal{E}_i) \rangle\rangle_2 - \langle\langle T(\mathcal{E}_j) T(\mathcal{F}_i) \rangle\rangle_2 = \frac{1}{2}T(\mathcal{E}_j)B(\mathcal{E}_i, \mathcal{E}_i, \mathcal{E}_i).$$

Applying equation (20) to the last term on the left hand side of the above equation, we obtain

$$\langle\langle \mathcal{E}_i T(\mathcal{E}_j) \rangle\rangle_2 = \langle\langle T(\mathcal{F}_j \circ \mathcal{E}_i) \rangle\rangle_2 + 3 \langle\langle T(\mathcal{E}_j \circ \mathcal{F}_i) \rangle\rangle_2 + A_2(\mathcal{F}_i, \mathcal{E}_j) + \frac{1}{2} T(\mathcal{E}_j) B(\mathcal{E}_i, \mathcal{E}_i, \mathcal{E}_i).$$

Plugging this formula into equation (22), we obtain

$$\begin{aligned} B(\mathcal{E}_i, \mathcal{E}_i, \mathcal{E}_j) &= -2 \langle\langle \mathcal{E}_i \mathcal{E}_i \mathcal{E}_j \gamma^\alpha \rangle\rangle_0 \langle\langle T(\gamma_\alpha) \rangle\rangle_2 + 2 \langle\langle T(\mathcal{F}_j \circ \mathcal{E}_i) \rangle\rangle_2 - 2 \langle\langle T(\mathcal{F}_i \circ \mathcal{E}_j) \rangle\rangle_2 \\ &\quad + A_2(\mathcal{F}_j, \mathcal{E}_i) + \frac{1}{2} T(\mathcal{E}_i) B(\mathcal{E}_j, \mathcal{E}_j, \mathcal{E}_j) - A_2(\mathcal{F}_i, \mathcal{E}_j) - \frac{1}{2} T(\mathcal{E}_j) B(\mathcal{E}_i, \mathcal{E}_i, \mathcal{E}_i). \end{aligned}$$

The first three terms on the right hand side of this equation are cancelled with each other due to equation (21) (iii). This proves (c).  $\square$

**Remark:** The reason that the general form of equation (16) gives no more information on the genus-2 generating function than what we can get from equation (19) lies behind the proof of this lemma.

**Corollary 2.6** *For any integers  $m \geq 0$  and  $k \geq 0$ ,*

$$2 \sum_{i=1}^N u_i^{m+k} B(\mathcal{E}_i, \mathcal{E}_i, \mathcal{E}_i) = \sum_{i=1}^N u_i^m B(\mathcal{E}_i, \mathcal{E}_i, \overline{\mathcal{X}}^k) + \sum_{i=1}^N u_i^k B(\mathcal{E}_i, \mathcal{E}_i, \overline{\mathcal{X}}^m).$$

**Proof:** Since  $T(\mathcal{W}) u_i = 0$  for any vector field  $\mathcal{W}$ , multiplying Lemma 2.5 (c) by  $u_i^m u_j^k$  and summing over  $i$  and  $j$ , we obtain

$$\begin{aligned} &\sum_{i=1}^N u_i^m B(\mathcal{E}_i, \mathcal{E}_i, \overline{\mathcal{X}}^k) - \sum_{i=1}^N u_i^{m+k} B(\mathcal{E}_i, \mathcal{E}_i, \mathcal{E}_i) \\ &= \frac{1}{2} T(\overline{\mathcal{X}}^m) \sum_{i=1}^N u_i^k B(\mathcal{E}_i, \mathcal{E}_i, \mathcal{E}_i) - \frac{1}{2} T(\overline{\mathcal{X}}^k) \sum_{i=1}^N u_i^m B(\mathcal{E}_i, \mathcal{E}_i, \mathcal{E}_i) \\ &\quad + A_2\left(\sum_{i=1}^N u_i^k \mathcal{F}_i, \overline{\mathcal{X}}^m\right) - A_2\left(\sum_{i=1}^N u_i^m \mathcal{F}_i, \overline{\mathcal{X}}^k\right). \end{aligned}$$

Observe that the right hand side of this equation is anti-symmetric with respect to  $m$  and  $k$ . So the desired formula is obtained by symmetrizing this equation with respect to  $m$  and  $k$ .  $\square$

### 3 Genus-2 Virasoro conjecture for manifolds with semisimple quantum cohomology

By [L2, Theorem 5.9], the genus-2 Virasoro conjecture for any smooth projective variety can be reduced to the genus-2  $L_1$ -constraint which have the following form:

$$\langle\langle \mathcal{L}_1 \rangle\rangle_2 = -\frac{1}{2} \sum_{\alpha} b_{\alpha} (1 - b_{\alpha}) \{ \langle\langle \gamma^{\alpha} \gamma_{\alpha} \rangle\rangle_1 + \langle\langle \gamma^{\alpha} \rangle\rangle_1 \langle\langle \gamma_{\alpha} \rangle\rangle_1 \}. \quad (23)$$

So to prove Theorem 0.2, it suffices to compute  $\langle\langle \mathcal{L}_1 \rangle\rangle_2 = \mathcal{L}_1 F_2$  and check whether it coincides with the right hand side of equation (23). There are two approaches to this problem: The first approach is to take the formula for  $F_2$  in Theorem 0.1 and then take the derivative along  $\mathcal{L}_1$ . The second approach is to directly use the formula for  $\langle\langle \mathcal{L}_1 \rangle\rangle_2$  given in Theorem 2.4. Since the intermediate results of the first approach provide more understanding for the genus-2 generating function and the Virasoro vector fields, we prove Theorem 0.2 using the first approach in this section. The second approach will be given in the appendix.

### 3.1 Express everything in terms of idempotents

Equation (23) is given in the flat frame  $\{\gamma_\alpha \mid \alpha = 1, \dots, N\}$ , so is the tensor  $A_1$  defined after equation (14). But idempotents has appeared in the expression of  $F_2$  as given in Theorem 0.1. To compare  $\mathcal{L}_1 F_2$  with the formula in equation (23), we need to re-write both of them using idempotents only. For this purpose, it is convenient to introduce the following notation:

$$z_{i_1, \dots, i_k} := \langle\langle \mathcal{E}_{i_1} \cdots \mathcal{E}_{i_k} \rangle\rangle_0$$

and

$$\phi_{i_1, \dots, i_k} := \langle\langle \mathcal{E}_{i_1} \cdots \mathcal{E}_{i_k} \rangle\rangle_1.$$

We will use the following simple fact which was explained in [L4]: For any tensor  $Q$ ,

$$Q(\gamma_\alpha, \gamma^\alpha, \dots) = \sum_{i=1}^N \frac{1}{g_i} Q(\mathcal{E}_i, \mathcal{E}_i, \dots). \quad (24)$$

In particular

$$\Delta := \gamma_\alpha \circ \gamma^\alpha = \sum_{i=1}^N \frac{1}{g_i} \mathcal{E}_i. \quad (25)$$

So the prediction of genus-2  $L_1$ -constraint, i.e. equation (23), can be re-written as

$$\langle\langle \mathcal{L}_1 \rangle\rangle_2 = -\frac{1}{2} \sum_{i=1}^N \frac{1}{g_i} \left\{ \langle\langle (\mathcal{G} * \mathcal{E}_i) (\mathcal{G} * \mathcal{E}_i) \rangle\rangle_1 + \langle\langle (\mathcal{G} * \mathcal{E}_i) \rangle\rangle_1^2 \right\}. \quad (26)$$

Using the definition of tensor  $B$  in [L2], we can write down the precise formula for the function  $B(\mathcal{E}_i, \mathcal{E}_i, \mathcal{E}_i)$ :

$$\begin{aligned} B(\mathcal{E}_i, \mathcal{E}_i, \mathcal{E}_i) &= \sum_{j,k} \frac{1}{g_j g_k} \left\{ \frac{1}{5} z_{iiijk} \phi_j \phi_k - \frac{6}{5} z_{iiij} \phi_{jk} \phi_k - \frac{6}{5} z_{iiijk} \phi_j \phi_{ik} \right\} \\ &+ \sum_j \frac{1}{g_j} \left\{ \frac{9}{5} (1 - 2\delta_{ij}) \phi_{ij}^2 - \frac{6}{5} (1 - 2\delta_{ij}) \phi_{ij} \phi_j \right\} \\ &+ \sum_{j,k} \frac{1}{g_j g_k} \left\{ \frac{1}{120} z_{iiijjk} \phi_k + \frac{1}{10} z_{iiijk} \phi_{jk} - \frac{1}{20} z_{iiij} \phi_{jkk} \right\} \end{aligned}$$

$$\begin{aligned}
& -\frac{3}{40}z_{iijjk}\phi_{ik} + \frac{3}{40}z_{ijjk}\phi_{iik} - \frac{3}{10}z_{iijk}\phi_{ijk} \Big\} \\
& - \sum_j \frac{1}{g_j} \left\{ \frac{1}{120}\phi_{iiij} + \frac{1}{20}(1-2\delta_{ij})\phi_{iijj} \right\}. \tag{27}
\end{aligned}$$

To express  $A_1$  in terms of idempotents, we will use decomposition (11) and derivatives of genus-0 and genus-1 topological recursion relations. Recall that the genus-0 topological recursion relation has the following form

$$\langle\langle T(\mathcal{W}) \mathcal{V}_1 \mathcal{V}_2 \rangle\rangle_0 = 0$$

for any vector fields  $\mathcal{W}$  and  $\mathcal{V}_i$ . Repeatedly taking derivatives of this relation, we have

$$\begin{aligned}
& \langle\langle T(\mathcal{W}) \mathcal{V}_1 \cdots \mathcal{V}_{k+2} \rangle\rangle_0 \\
& = \sum_{m=1}^k \sum_{1 \leq i_1 < \cdots < i_m \leq k} \langle\langle \mathcal{W} \mathcal{V}_{i_1} \cdots \mathcal{V}_{i_m} \gamma^\alpha \rangle\rangle_0 \\
& \quad \cdot \langle\langle \gamma_\alpha \mathcal{V}_1 \cdots \widehat{\mathcal{V}_{i_1}} \cdots \widehat{\mathcal{V}_{i_2}} \cdots \cdots \cdots \widehat{\mathcal{V}_{i_m}} \cdots \mathcal{V}_{k+2} \rangle\rangle_0. \tag{28}
\end{aligned}$$

The genus-1 topological recursion relation has the form

$$\langle\langle T(\mathcal{W}) \rangle\rangle_1 = \frac{1}{24} \langle\langle \mathcal{W} \gamma_\alpha \gamma^\alpha \rangle\rangle_0$$

for any vector field  $\mathcal{W}$ . Repeatedly taking derivatives of this relation, we obtain

$$\begin{aligned}
& \langle\langle T(\mathcal{W}) \mathcal{V}_1 \cdots \mathcal{V}_k \rangle\rangle_1 \\
& = \sum_{m=1}^k \sum_{1 \leq i_1 < \cdots < i_m \leq k} \langle\langle \mathcal{W} \mathcal{V}_{i_1} \cdots \mathcal{V}_{i_m} \gamma^\alpha \rangle\rangle_0 \\
& \quad \cdot \langle\langle \gamma_\alpha \mathcal{V}_1 \cdots \widehat{\mathcal{V}_{i_1}} \cdots \widehat{\mathcal{V}_{i_2}} \cdots \cdots \cdots \widehat{\mathcal{V}_{i_m}} \cdots \mathcal{V}_k \rangle\rangle_1 \\
& \quad + \frac{1}{24} \langle\langle \mathcal{W} \mathcal{V}_1 \cdots \mathcal{V}_k \gamma^\alpha \gamma_\alpha \rangle\rangle_0 \tag{29}
\end{aligned}$$

for all vector fields  $\mathcal{W}$  and  $\mathcal{V}_i$ , and all integer  $k \geq 0$ . For any vector field  $\mathcal{W}$ , first decomposing it as

$$\mathcal{W} = \overline{\mathcal{W}} + T(\overline{\tau_-(\mathcal{W})}) + T^2(\tau_-^2(\mathcal{W})),$$

then using equation (28) and (29) to get rid of the operator  $T$ , we obtain

$$\begin{aligned}
A_1(\mathcal{W}) &= \frac{7}{10} \langle\langle \gamma^\alpha \rangle\rangle_1 \langle\langle (\gamma_\alpha \circ \overline{\mathcal{W}}) \rangle\rangle_1 \\
&+ \frac{1}{10} \langle\langle \gamma^\alpha (\gamma_\alpha \circ \overline{\mathcal{W}}) \rangle\rangle_1 - \frac{1}{240} \langle\langle \overline{\mathcal{W}} \Delta \rangle\rangle_1 \\
&+ \frac{1}{20} \langle\langle \{\overline{\tau_-(\mathcal{W})} \circ \Delta\} \rangle\rangle_1 + \frac{13}{240} \langle\langle \overline{\mathcal{W}} \gamma_\alpha \gamma^\alpha \gamma^\beta \rangle\rangle_0 \langle\langle \gamma_\beta \rangle\rangle_1 \\
&+ \frac{1}{1152} \langle\langle \mathcal{S} \mathcal{S} \{\overline{\tau_-^2(\mathcal{W})} \circ \Delta^2\} \rangle\rangle_0 + \frac{1}{1152} \langle\langle \overline{\tau_-(\mathcal{W})} \Delta \gamma^\alpha \gamma_\alpha \rangle\rangle_0 \\
&+ \frac{1}{480} \langle\langle \{\overline{\tau_-(\mathcal{W})} \circ \gamma_\alpha\} \gamma^\alpha \gamma_\beta \gamma^\beta \rangle\rangle_0 + \frac{1}{960} \langle\langle \overline{\mathcal{W}} \gamma_\alpha \gamma^\alpha \gamma_\beta \gamma^\beta \rangle\rangle_0.
\end{aligned}$$

In semisimple case, we can use idempotents to express this tensor as

$$\begin{aligned}
A_1(\mathcal{W}) = & \sum_i \frac{\langle \mathcal{W}, \mathcal{E}_i \rangle}{g_i} \left\{ \frac{1}{g_i} \left( \frac{7}{10} \phi_i^2 + \frac{1}{10} \phi_{ii} \right) - \sum_j \frac{1}{240} \frac{1}{g_j} \phi_{ij} \right. \\
& \left. + \sum_{j,k} \left( \frac{13}{240} \frac{1}{g_j g_k} z_{ijjk} \phi_k + \frac{1}{960} \frac{1}{g_j g_k} z_{ijjkk} \right) \right\} \\
& + \sum_i \frac{\langle \tau_-(\mathcal{W}), \mathcal{E}_i \rangle}{g_i} \left\{ \frac{1}{20} \frac{1}{g_i} \phi_i + \sum_j \frac{1}{480} \frac{1}{g_i g_j} z_{iijj} + \sum_{j,k} \frac{1}{1152} \frac{1}{g_j g_k} z_{ijkk} \right\} \\
& + \sum_i \frac{\langle \tau_-^2(\mathcal{W}), \mathcal{E}_i \rangle}{g_i} \cdot \frac{1}{1152} \frac{1}{g_i}. \tag{30}
\end{aligned}$$

### 3.2 Expressing everything by rotation coefficients

Relations among functions  $z_{i_1, \dots, i_k}$  and  $\phi_{i_1, \dots, i_k}$  are very complicated. It's much easier to see the relations by introducing rotation coefficients. On the small phase space, rotation coefficients was introduced by Dubrovin [D] to study semisimple Frobenius manifolds. A similar definition for *rotation coefficients* on the big phase space is the following:

$$r_{ij} := \frac{\mathcal{E}_j}{\sqrt{g_j}} \sqrt{g_i}.$$

We will briefly review basic properties of rotation coefficients when they are needed. The readers are referred to [L4] for more details. First,  $(r_{ij})$  is a symmetric matrix. Using these functions, the operator  $\mathcal{G}^*$  is given by

$$\mathcal{G}^* \mathcal{E}_i = \frac{1}{2} \mathcal{E}_i + \sum_j (u_i - u_j) r_{ij} \sqrt{\frac{g_i}{g_j}} \mathcal{E}_j \tag{31}$$

for all  $i$ . Therefore, the prediction of the genus-2  $L_1$ -constraint, i.e. equation (26), is given by

$$\begin{aligned}
\langle\langle \mathcal{L}_1 \rangle\rangle_2 = & -\frac{1}{2} \sum_{i=1}^N \frac{1}{g_i} \left\{ \frac{1}{4} (\phi_{ii} + \phi_i^2) + \sum_j (u_i - u_j) r_{ij} \sqrt{\frac{g_i}{g_j}} (\phi_{ij} + \phi_i \phi_j) \right. \\
& \left. + \sum_{j,k} (u_i - u_j) (u_i - u_k) r_{ij} r_{ik} \frac{g_i}{\sqrt{g_j g_k}} (\phi_{jk} + \phi_j \phi_k) \right\}.
\end{aligned}$$

Note that the second term is anti-symmetric with respect to  $i$  and  $j$ , so equal to 0 when summing over  $i$  and  $j$ . Define

$$v_{ij} := (u_j - u_i) r_{ij}.$$

Then  $v_{ij} = -v_{ji}$  for all  $i$  and  $j$ . The prediction of the genus-2  $L_1$ -constraint can now be written as

$$\langle\langle \mathcal{L}_1 \rangle\rangle_2 = -\frac{1}{2} \sum_{i=1}^N \frac{1}{g_i} \left\{ \frac{1}{4} (\phi_{ii} + \phi_i^2) + \sum_{j,k} v_{ij} v_{ik} \frac{g_i}{\sqrt{g_j g_k}} (\phi_{jk} + \phi_j \phi_k) \right\}. \tag{32}$$

Covariant derivatives of idempotents are given by

$$\nabla_{\mathcal{E}_i} \mathcal{E}_j = r_{ij} \left( \sqrt{\frac{g_j}{g_i}} \mathcal{E}_i + \sqrt{\frac{g_i}{g_j}} \mathcal{E}_j \right) - \delta_{ij} \sum_{k=1}^N r_{ik} \sqrt{\frac{g_i}{g_k}} \mathcal{E}_k \quad (33)$$

for any  $i$  and  $j$ . To compute the derivatives of rotation coefficients, we define

$$\theta_{ij} := \frac{1}{u_j - u_i} \left( r_{ij} + \sum_k r_{ik} v_{jk} \right)$$

for  $i \neq j$ . These functions satisfy the following property

$$\theta_{ij} + \theta_{ji} = - \sum_k r_{ik} r_{jk} \quad (34)$$

for any  $i \neq j$ . First derivatives of rotation coefficients are given by the formula

$$\mathcal{E}_k r_{ij} = r_{ik} r_{jk} + \begin{cases} 0, & \text{if } i \neq j \neq k, \\ \theta_{ij}, & \text{if } k = i \neq j, \\ \sqrt{\frac{g_k}{g_i}} \theta_{ik}, & \text{if } i = j \neq k, \\ -2 \sum_l r_{il}^2 + \sum_{p \neq i} \sqrt{\frac{g_p}{g_i}} \theta_{pi} + \frac{1}{g_i} \langle \tau_-^2(\mathcal{S}), \mathcal{E}_i \rangle, & \text{if } i = j = k. \end{cases} \quad (35)$$

The appearance of  $\langle \tau_-^2(\mathcal{S}), \mathcal{E}_i \rangle$  in the last equation is a typical big phase space phenomenon. This term vanishes on the small phase space, but is in general not zero on the big phase space. To compute higher order derivatives of rotation coefficients, we also need the following formula:

$$\mathcal{E}_j \langle \tau_-^k(\mathcal{S}), \mathcal{E}_i \rangle = \delta_{ij} \langle \tau_-^{k+1}(\mathcal{S}), \mathcal{E}_i \rangle + \langle \tau_-^k(\mathcal{S}), \nabla_{\mathcal{E}_j} \mathcal{E}_i \rangle \quad (36)$$

for all  $k \geq 0$ . This formula follows from equation (9). We can repeatedly apply the above formulas to compute higher order derivatives of rotation coefficients in terms of functions  $g_i$ ,  $u_i$ ,  $r_{ij}$ , and  $\langle \tau_-^k(\mathcal{S}), \mathcal{E}_i \rangle$  with  $k \geq 2$ . When we compute  $k$ -th order derivatives of  $r_{ij}$ , we might encounter  $\langle \tau_-^{k+1}(\mathcal{S}), \mathcal{E}_i \rangle$ . For most purposes (for example, for the proof of genus-1 and genus-2 Virasoro conjecture) these terms will not affect the final results. In this paper, we only need derivatives of rotation coefficients up to order 3. It will be convenient to introduce the following functions when computing the second and third order derivatives:

$$\begin{aligned} \Omega_{ij} &:= \frac{1}{u_j - u_i} \left\{ \theta_{ij} - \theta_{ji} + \sum_{k,l} r_{il} r_{jk} v_{kl} \right\}, \\ \Lambda_{ij} &:= \frac{1}{u_j - u_i} \left\{ 3\Omega_{ij} - \sum_k (u_k - u_j) \theta_{ik} \theta_{jk} - \left( r_{ii} + \sum_k r_{ik} v_{ik} \right) \theta_{ji} \right. \\ &\quad \left. - \left( r_{jj} + \sum_k r_{jk} v_{jk} \right) \theta_{ij} \right\} \end{aligned}$$

for  $i \neq j$ . These functions have the property

$$\Omega_{ij} = \Omega_{ji} \quad \text{and} \quad \Lambda_{ij} + \Lambda_{ji} = \sum_k \theta_{ik} \theta_{jk}$$

for all  $i \neq j$ . They arise naturally in the second and third order derivatives of rotation coefficients because

$$\mathcal{E}_j \theta_{ij} = \left( r_{jj} - \sqrt{\frac{g_j}{g_i}} r_{ij} \right) \theta_{ij} - \Omega_{ij}$$

and

$$\mathcal{E}_i \Omega_{ij} = \theta_{ji} \left( \sum_k r_{ik}^2 + \sqrt{\frac{g_i}{g_j}} \theta_{ij} + \sum_{k \neq i} \sqrt{\frac{g_k}{g_i}} \theta_{ki} \right) + \Lambda_{ij}$$

for  $i \neq j$ . We might consider  $\theta_{ij}$ ,  $\Omega_{ij}$ , and  $\Lambda_{ij}$  as functions having poles of order 1, 2, and 3 respectively in terms of  $u_1, \dots, u_n$ .

We can assign  $g_i$  with degree 0,  $u_i$  with degree  $-1$ , and  $k$ -th order derivatives of  $r_{ij}$  along directions of idempotents with degree  $k+1$ . Then most expressions in this paper are homogeneous of a fixed degree. For example, functions  $z_{i_1, \dots, i_k}$  have degree  $k-3$  for  $k \geq 3$ ,  $\phi_{i_1, \dots, i_k}$  have degree  $k$ ,  $\langle \tau_-^k(\mathcal{S}), \mathcal{E}_i \rangle$  has degree  $k$ ,  $\langle \tau_-^k(\mathcal{L}_m), \mathcal{E}_i \rangle$  has degree  $k-m-1$ ,  $B(\mathcal{E}_i, \mathcal{E}_i, \mathcal{E}_i)$  has degree 4,  $\langle \mathcal{L}_1 \rangle_2$  has degree 2, and  $F_2$  has degree 3. In general, we expect  $F_g$  to have degree  $3(g-1)$  for all  $g$ .

To express  $F_2$  in terms of rotation coefficients, we need first to use rotation coefficients to describe vector fields  $\overline{\tau_-^k(\mathcal{L}_0)}$ . A recursion formula for  $\overline{\tau_-^k(\mathcal{L}_m)}$  was given in [L2, Theorem 4.8]. In the semisimple case, using equations (7), (12) and (31), this recursion relation can be written as

$$\begin{aligned} \langle \tau_-^{k+1}(\mathcal{L}_m), \mathcal{E}_i \rangle &= u_i \langle \tau_-^{k+1}(\mathcal{L}_{m-1}), \mathcal{E}_i \rangle + \left(k + \frac{3}{2}\right) \langle \tau_-^k(\mathcal{L}_{m-1}), \mathcal{E}_i \rangle \\ &\quad + \sum_j v_{ij} \sqrt{\frac{g_i}{g_j}} \langle \tau_-^k(\mathcal{L}_{m-1}), \mathcal{E}_j \rangle. \end{aligned} \quad (37)$$

In particular, since  $\mathcal{L}_{-1} = -\mathcal{S}$ , we have

$$\begin{aligned} \langle \tau_-^k(\mathcal{L}_0), \mathcal{E}_i \rangle &= -u_i \langle \tau_-^k(\mathcal{S}), \mathcal{E}_i \rangle - \frac{2k+1}{2} \langle \tau_-^{k-1}(\mathcal{S}), \mathcal{E}_i \rangle \\ &\quad - \sum_j v_{ij} \sqrt{\frac{g_i}{g_j}} \langle \tau_-^{k-1}(\mathcal{S}), \mathcal{E}_j \rangle \end{aligned} \quad (38)$$

for  $k \geq 1$ . We will keep  $\langle \tau_-^k(\mathcal{S}), \mathcal{E}_i \rangle$  with  $k \geq 2$  in our computations. But for  $k=0$  and  $k=1$ , we will use (cf. [L4]),

$$\langle \mathcal{S}, \mathcal{E}_i \rangle = g_i \quad \text{and} \quad \langle \tau_-(\mathcal{S}), \mathcal{E}_i \rangle = \sum_j r_{ij} \sqrt{g_i g_j}. \quad (39)$$

We now describe how to represent  $z_{i_1, \dots, i_k}$  and  $\phi_{i_1, \dots, i_k}$  in terms of rotation coefficients. In [L4], it was proved that genus-0 4-point functions have the following property: For

$i \neq j$ ,

$$\begin{aligned}
(i) \quad & z_{iiii} = -g_i r_{ii}, \\
(ii) \quad & z_{jiii} = -z_{jjii} = -\sqrt{g_i g_j} r_{ij}, \\
(iii) \quad & z_{ijkl} = 0 \quad \text{otherwise.}
\end{aligned} \tag{40}$$

It was also proved that genus-1 1-point functions are given by

$$24\phi_i = -12 \sum_j r_{ij} v_{ij} - \sum_j \sqrt{\frac{g_i}{g_j}} r_{ij}. \tag{41}$$

Note that for any positive integer  $k$ ,

$$\begin{aligned}
\langle\langle \mathcal{E}_{i_1} \mathcal{E}_{i_2} \cdots \mathcal{E}_{i_{k+1}} \rangle\rangle_g &= \mathcal{E}_{i_{k+1}} \langle\langle \mathcal{E}_{i_1} \cdots \mathcal{E}_{i_k} \rangle\rangle_g - \sum_{j=1}^k \langle\langle \mathcal{E}_{i_1} \cdots (\nabla_{\mathcal{E}_{i_{k+1}}} \mathcal{E}_{i_j}) \cdots \mathcal{E}_{i_k} \rangle\rangle_g \\
&= \mathcal{E}_{i_{k+1}} \langle\langle \mathcal{E}_{i_1} \cdots \mathcal{E}_{i_k} \rangle\rangle_g - \left( \sum_{j=1}^k r_{i_j, i_{k+1}} \sqrt{\frac{g_{i_{k+1}}}{g_{i_j}}} \right) \langle\langle \mathcal{E}_{i_1} \cdots \mathcal{E}_{i_k} \rangle\rangle_g \\
&\quad - \sum_{j=1}^k r_{i_j, i_{k+1}} \sqrt{\frac{g_{i_j}}{g_{i_{k+1}}}} \langle\langle \mathcal{E}_{i_1} \cdots \widehat{\mathcal{E}_{i_j}} \cdots \mathcal{E}_{i_k} \mathcal{E}_{i_{k+1}} \rangle\rangle_g \\
&\quad + \sum_{j=1}^k \delta_{i_{k+1}, i_j} \sum_p r_{p, i_{k+1}} \sqrt{\frac{g_{i_{k+1}}}{g_p}} \langle\langle \mathcal{E}_{i_1} \cdots \widehat{\mathcal{E}_{i_j}} \cdots \mathcal{E}_{i_k} \mathcal{E}_p \rangle\rangle_g. \tag{42}
\end{aligned}$$

Repeatedly using this formula, we can express all functions  $z_{i_1 \dots i_k}$ , with  $k \geq 4$ , and  $\phi_{i_1 \dots i_k}$ , with  $k \geq 1$ , in terms of rotation coefficients. For example, genus-1 2-point functions are given by

$$\begin{aligned}
24\phi_{ii} &= 12r_{ii}^2 - \frac{1}{g_i} \langle \tau_-^2(\mathcal{S}), \mathcal{E}_i \rangle + \sum_j \left\{ r_{ij}^2 \left( -10 + \frac{g_i}{g_j} \right) + 24r_{ii} r_{ij} v_{ij} \right\} \\
&\quad - \sum_{j,k} \left( 12r_{ij} r_{jk} v_{jk} \sqrt{\frac{g_i}{g_j}} + r_{ij} r_{jk} \sqrt{\frac{g_i}{g_k}} \right) - \sum_{j \neq i} \left( \theta_{ij} \sqrt{\frac{g_i}{g_j}} + \theta_{ji} \sqrt{\frac{g_j}{g_i}} \right) \tag{43}
\end{aligned}$$

for all  $i$ , and

$$\begin{aligned}
24\phi_{ij} &= 12r_{ij}^2 + \sum_k \left\{ r_{ik} r_{jk} \frac{\sqrt{g_i g_j}}{g_k} + 12r_{ij} r_{ik} v_{ik} \sqrt{\frac{g_j}{g_i}} + 12r_{ij} r_{jk} v_{jk} \sqrt{\frac{g_i}{g_j}} \right\} \\
&\quad - \left( \theta_{ij} \sqrt{\frac{g_j}{g_i}} + \theta_{ji} \sqrt{\frac{g_i}{g_j}} \right) \tag{44}
\end{aligned}$$

for  $i \neq j$ . When  $k$  becomes larger, the formula for  $z_{i_1 \dots i_k}$  and  $\phi_{i_1 \dots i_k}$  becomes more complicated. It is not illuminating to write them out here. But one should notice that to express  $F_2$  in terms of rotation coefficients, we only need  $z_{i_1 \dots i_k}$  for  $4 \leq k \leq 6$  and  $\phi_{i_1 \dots i_k}$  for  $1 \leq k \leq 4$ , which can be obtained by taking derivaties of equations (40), (43), (44) twice. Combining with the formula in Theorem 0.1, and equations (27), (30), (38), (39), a lengthy but straightforward computation shows the following:

**Theorem 3.1** *For any manifold with semisimple quantum cohomology, the genus-2 generating function for the Gromov-Witten invariants is given by the following formula:*

$$\begin{aligned}
& 5760 F_2 \\
&= -5 \sum_i \frac{1}{g_i^2} \langle \tau_-^3(\mathcal{S}), \mathcal{E}_i \rangle + \sum_i \sum_{j \neq i} 5\Omega_{ij} \left( \frac{1}{g_j} \sqrt{\frac{g_i}{g_j}} - \frac{1}{\sqrt{g_i g_j}} \right) \\
&+ \sum_i \frac{1}{g_i} \langle \tau_-^2(\mathcal{S}), \mathcal{E}_i \rangle \left\{ 24r_{ii} \frac{1}{g_i} + \sum_j \left( 5r_{ij} \frac{1}{g_j} \sqrt{\frac{g_i}{g_j}} + 144r_{ij}v_{ij} \frac{1}{g_i} \right) \right\} \\
&+ \sum_i \sum_{j \neq i} \theta_{ij} \left\{ -24r_{ii} \frac{1}{g_i} \sqrt{\frac{g_j}{g_i}} + 200r_{ij} \frac{1}{g_j} \right. \\
&\quad \left. + \sum_k \left[ r_{ik}v_{ik} \left( 120 \frac{1}{\sqrt{g_i g_j}} - 144 \frac{1}{g_i} \sqrt{\frac{g_j}{g_i}} \right) + r_{jk}v_{ik} \left( 85 \frac{1}{g_i} + 45 \frac{1}{g_j} \right) \right] \right\} \\
&- \sum_i 576r_{ii}^3 \frac{1}{g_i} - \sum_i 576 \frac{1}{g_i} \left( \sum_j r_{ij}v_{ij} \right)^3 \\
&+ \sum_{i,j} \left\{ 480r_{ij}^3 \frac{1}{\sqrt{g_i g_j}} - 23r_{ii}r_{ij}^2 \frac{1}{g_i} - 1728r_{ii}^2 r_{ij}v_{ij} \frac{1}{g_i} \right\} \\
&+ \sum_{i,j,k} \left\{ -24r_{ii}r_{ik}r_{jk} \frac{1}{g_i} \sqrt{\frac{g_j}{g_i}} + 115r_{ij}r_{ik}r_{jk} \frac{1}{g_i} \right. \\
&\quad \left. + 1452r_{ik}^2(r_{ij}v_{ij}) \frac{1}{g_i} - 1728r_{ii} \frac{1}{g_i} (r_{ij}v_{ij})(r_{ik}v_{ik}) \right\} \\
&+ \sum_{i,j,k,l} \left\{ 120r_{ik}r_{jk}(r_{il}v_{il}) \frac{1}{\sqrt{g_i g_j}} - 144r_{ij}r_{il}(r_{jk}v_{jk}) \frac{1}{g_j} \sqrt{\frac{g_l}{g_j}} \right. \\
&\quad \left. - 40r_{ik}r_{jk}r_{il}v_{jl} \frac{1}{g_i} + 720r_{ij}(r_{ik}v_{ik})(r_{jl}v_{jl}) \frac{1}{\sqrt{g_i g_j}} \right\}.
\end{aligned}$$

This formula is much simpler than the formulas for expressing 4-point genus-1 functions  $\phi_{iiij}$  and  $\phi_{iijj}$  in terms of rotation coefficients (which were omitted here). We also note that the right hand side of this formula only depends on genus-0 data. In [DZ3, p157-160], a three-page formula of  $F_2$  for semisimple Frobenius manifolds was derived under the assumption that  $F_2$  satisfies the Virasoro constraints. Comparatively, Theorem 3.1 gives a much simpler formula for  $F_2$ .

Note that when computing 4-point genus-1 functions  $\phi_{iiij}$  and  $\phi_{iijj}$  using equations (41) and (42), we will encounter third order derivatives of rotation coefficients. This causes that  $B(\mathcal{E}_i, \mathcal{E}_i, \mathcal{E}_i)$  contains third order poles of the form

$$\frac{1}{576} \sum_{j \neq i} \left\{ \Lambda_{ji} \left( \frac{1}{\sqrt{g_i g_j}} - \frac{1}{g_j} \sqrt{\frac{g_i}{g_j}} \right) - \Lambda_{ij} \left( \frac{1}{\sqrt{g_i g_j}} - \frac{1}{g_i} \sqrt{\frac{g_j}{g_i}} \right) \right\}.$$

When interchanging  $i$  and  $j$ , the second term in this expression is precisely the first term

with an opposite sign. Multiplying this expression by  $u_i$  and summing over  $i$ , we obtain

$$\frac{1}{576} \sum_i \sum_{j \neq i} (u_i - u_j) \Lambda_{ji} \left( \frac{1}{\sqrt{g_i g_j}} - \frac{1}{g_j} \sqrt{\frac{g_i}{g_j}} \right).$$

By the definition of  $\Lambda_{ij}$ ,  $(u_i - u_j) \Lambda_{ji}$  only has second order poles. Therefore third order poles do not appear in the formula for  $F_2$  in Theorem 3.1. Similar observations have also been used to simply terms with first and second order poles when computing  $F_2$ .

Also note that the 4-point genus-1 functions  $\phi_{iiii}$  will produce a term  $-\frac{1}{576} \langle \tau_-^4(\mathcal{S}), \mathcal{E}_i \rangle$  in  $B(\mathcal{E}_i, \mathcal{E}_i, \mathcal{E}_i)$ . After multiplied by  $u_i$  and summed over  $i$ , this term will be cancelled with the corresponding term produced by  $2A_1(\tau_-^2(\mathcal{L}_0))$ . Therefore the formula for  $F_2$  in Theorem 3.1 only contains  $\langle \tau_-^k(\mathcal{S}), \mathcal{E}_i \rangle$  for  $k = 2, 3$ .

### 3.3 Action of $\mathcal{L}_n$

In this section, we discuss the action of Virasoro vector fields on functions  $u_i$ ,  $g_i$ ,  $r_{ij}$ , and  $\langle \tau_-^k(\mathcal{S}), \mathcal{E}_i \rangle$ . Although for the proof of Theorem 0.2 we only need to know the action of  $\mathcal{L}_1$ , we will discuss the action for all  $\mathcal{L}_m$  since they may be needed for the study of higher genus Gromov-Witten invariants.

Recall for any vector field  $\mathcal{W}$ , we have the standard decomposition

$$\mathcal{W} = \overline{\mathcal{W}} + T(\tau_-(\mathcal{W})).$$

In previous sections, we have given formulas for the action of  $\mathcal{E}_i$  on functions  $u_i$ ,  $g_i$ ,  $r_{ij}$ , and  $\langle \tau_-^k(\mathcal{S}), \mathcal{E}_i \rangle$ . Since for any vector field  $\mathcal{W}$ , its primary projection  $\overline{\mathcal{W}}$  has the decomposition

$$\overline{\mathcal{W}} = \sum_i \frac{\langle \mathcal{W}, \mathcal{E}_i \rangle}{g_i} \mathcal{E}_i,$$

the action of  $\overline{\mathcal{W}}$  is thus well understood. In [L4], we have shown the following formula

$$\begin{aligned} T(\mathcal{W})u_i &= 0, \\ T(\mathcal{W})g_i &= -2 \langle \mathcal{W}, \mathcal{E}_i \rangle, \\ T(\mathcal{W})r_{ij} &= \delta_{ij} \left\{ -\frac{\langle \tau_-(\mathcal{W}), \mathcal{E}_i \rangle}{g_i} + \sum_k r_{ik} \frac{1}{\sqrt{g_i g_k}} \langle \mathcal{W}, \mathcal{E}_k \rangle \right\} \end{aligned}$$

for any vector field  $\mathcal{W}$ . Therefore for any vector field  $\mathcal{W}$ , we have the followings formula

$$\begin{aligned} \mathcal{W}u_i &= \frac{1}{g_i} \langle \mathcal{W}, \mathcal{E}_i \rangle, \\ \mathcal{W}g_i &= -2 \langle \tau_-(\mathcal{W}), \mathcal{E}_i \rangle + \sum_j 2r_{ij} \sqrt{\frac{g_i}{g_j}} \langle \mathcal{W}, \mathcal{E}_j \rangle, \\ \mathcal{W}r_{ij} &= \frac{1}{g_i} \langle \mathcal{W}, \mathcal{E}_i \rangle \theta_{ij} + \frac{1}{g_j} \langle \mathcal{W}, \mathcal{E}_j \rangle \theta_{ji} + \sum_k \frac{1}{g_k} \langle \mathcal{W}, \mathcal{E}_k \rangle r_{ik} r_{jk} \quad \text{for } i \neq j, \end{aligned}$$

$$\begin{aligned}
\mathcal{W}r_{ii} = & \sum_{j \neq i} \left\{ \frac{1}{g_j} \langle \mathcal{W}, \mathcal{E}_j \rangle \theta_{ij} + \frac{1}{g_i} \langle \mathcal{W}, \mathcal{E}_i \rangle \theta_{ji} \right\} \sqrt{\frac{g_j}{g_i}} \\
& - \frac{1}{g_i} \langle \tau_-^2(\mathcal{W}), \mathcal{E}_i \rangle + \frac{1}{g_i^2} \langle \mathcal{W}, \mathcal{E}_i \rangle \langle \tau_-^2(\mathcal{S}), \mathcal{E}_i \rangle + \sum_j \frac{r_{ij}}{\sqrt{g_i g_j}} \langle \tau_-(\mathcal{W}), \mathcal{E}_j \rangle \\
& + \sum_j r_{ij}^2 \left\{ \frac{1}{g_j} \langle \mathcal{W}, \mathcal{E}_j \rangle - 2 \frac{1}{g_i} \langle \mathcal{W}, \mathcal{E}_i \rangle \right\}.
\end{aligned} \tag{45}$$

For the Virasoro vector fields  $\mathcal{L}_m$ , we have

$$\langle \mathcal{L}_m, \mathcal{E}_i \rangle = -u_i^{m+1} g_i$$

since  $\overline{\mathcal{L}}_m = -\overline{\mathcal{X}}^{m+1}$ . The recursion formula (37) implies that

$$\langle \tau_-(\mathcal{L}_m), \mathcal{E}_i \rangle = -\frac{3}{2}(m+1)u_i^m g_i - \sum_j u_j^{m+1} r_{ij} \sqrt{g_i g_j} \tag{46}$$

and

$$\begin{aligned}
\langle \tau_-^2(\mathcal{L}_m), \mathcal{E}_i \rangle = & -u_i^{m+1} \langle \tau_-^2(\mathcal{S}), \mathcal{E}_i \rangle - \frac{1}{8} m(11m+19) u_i^{m-1} g_i \\
& - \sum_j r_{ij} \sqrt{g_i g_j} \left( 2m u_i^m + \frac{1}{2} (3m+7) u_j^m - \sum_{p=0}^m u_i^p u_j^{m-p} \right) \\
& - \sum_{j,k} v_{ij} r_{jk} \sqrt{g_i g_k} \sum_{p=0}^m u_i^p u_k^{m-p}.
\end{aligned} \tag{47}$$

These formulas enable us to compute  $\mathcal{L}_m u_i$ ,  $\mathcal{L}_m g_i$ , and  $\mathcal{L}_m r_{ij}$ . When computing  $\mathcal{L}_m r_{ij}$ , we will encounter first order poles of the type  $u_i^{m+1} \theta_{ij} + u_j^{m+1} \theta_{ji}$ . By equation (34), we have

$$u_i^{m+1} \theta_{ij} + u_j^m \theta_{ji} = (u_i^{m+1} - u_j^{m+1}) \theta_{ij} - u_j^{m+1} \sum_k r_{ik} r_{jk}.$$

Factoring out  $u_i - u_j$  from  $u_i^{m+1} - u_j^{m+1}$  and using the definition of  $\theta_{ij}$ , we have

$$u_i^{m+1} \theta_{ij} + u_j^{m+1} \theta_{ji} = -r_{ij} \sum_{p=0}^m u_i^p u_j^{m-p} - \sum_k r_{ik} r_{jk} \left\{ u_j^{m+1} + (u_k - u_j) \sum_{p=0}^m u_i^p u_j^{m-p} \right\}.$$

Similarly,

$$u_j^{m+1} \theta_{ij} + u_i^{m+1} \theta_{ji} = r_{ij} \sum_{p=0}^m u_i^p u_j^{m-p} + \sum_k r_{ik} r_{jk} \left\{ -u_i^{m+1} + (u_k - u_j) \sum_{p=0}^m u_i^p u_j^{m-p} \right\}.$$

Therefore first order poles  $\theta_{ij}$  will not appear in  $\mathcal{L}_m r_{ij}$  for all  $i$  and  $j$ . A straightforward computation shows the following

**Lemma 3.2** For  $i \neq j$ ,

$$\begin{aligned}
\mathcal{L}_m u_i &= -u_i^{m+1}, \\
\mathcal{L}_m g_i &= 3(m+1)u_i^m g_i, \\
\mathcal{L}_m r_{ij} &= r_{ij} \sum_{p=0}^m u_i^p u_j^{m-p} + \sum_k r_{ik} r_{jk} \left\{ u_j^{m+1} - u_k^{m+1} + (u_k - u_j) \sum_{p=0}^m u_i^p u_j^{m-p} \right\}, \\
\mathcal{L}_m r_{ii} &= \frac{1}{8}m(11m+19)u_i^{m-1} + (m+1)u_i^m r_{ii} \\
&\quad + \sum_k r_{ik} \sqrt{\frac{g_k}{g_i}} \left( 2m u_i^m - \sum_{p=1}^m 2u_i^p u_k^{m-p} \right) \\
&\quad + \sum_k r_{ik}^2 \left\{ (m+1)u_i^m u_k - m u_i^{m+1} - u_k^{m+1} \right\}.
\end{aligned}$$

From this lemma, we see that the action of  $\mathcal{L}_m$  decreases the degree by  $m$ .

We now consider  $\mathcal{L}_m \langle \tau_-^k(\mathcal{S}), \mathcal{E}_i \rangle$ . We first note that for any vector field  $\mathcal{W}$ ,

$$\nabla_{\mathcal{W}} \tau_-^k(\mathcal{S}) = \tau_-^k(\nabla_{\mathcal{W}} \mathcal{S}) = -\tau_-^{k+1}(\mathcal{W})$$

for any  $k \geq 0$ . So by equation (9),

$$\mathcal{W} \langle \tau_-^k(\mathcal{S}), \mathcal{E}_i \rangle = -\langle \tau_-^{k+1}(\mathcal{W}), \mathcal{E}_i \rangle + \langle \tau_-^{k+1}(\mathcal{S}), \mathcal{W} \circ \mathcal{E}_i \rangle + \langle \tau_-^k(\mathcal{S}), \nabla_{\mathcal{W}} \mathcal{E}_i \rangle. \quad (48)$$

For the Virasoro vector field  $\mathcal{L}_m$ , using the standard decomposition, we have

$$\nabla_{\mathcal{L}_m} \mathcal{E}_i = -\nabla_{\overline{\mathcal{X}}^{m+1}} \mathcal{E}_i + \nabla_{T(\tau_-(\mathcal{L}_m))} \mathcal{E}_i = -\sum_j u_j^{m+1} \nabla_{\mathcal{E}_j} \mathcal{E}_i - \tau_-(\mathcal{L}_m) \circ \mathcal{E}_i.$$

By equations (33) and (46), we obtain the formula

$$\nabla_{\mathcal{L}_m} \mathcal{E}_i = \frac{3}{2}(m+1)u_i^m \mathcal{E}_i + \sum_j (u_i^{m+1} - u_j^{m+1}) r_{ij} \sqrt{\frac{g_i}{g_j}} \mathcal{E}_j \quad (49)$$

So by equation (48), we have

$$\begin{aligned}
\mathcal{L}_m \langle \tau_-^k(\mathcal{S}), \mathcal{E}_i \rangle &= -\langle \tau_-^{k+1}(\mathcal{L}_m), \mathcal{E}_i \rangle - u_i^{m+1} \langle \tau_-^{k+1}(\mathcal{S}), \mathcal{E}_i \rangle + \frac{3}{2}(m+1)u_i^m \langle \tau_-^k(\mathcal{S}), \mathcal{E}_i \rangle \\
&\quad + \sum_j (u_i^{m+1} - u_j^{m+1}) r_{ij} \sqrt{\frac{g_i}{g_j}} \langle \tau_-^k(\mathcal{S}), \mathcal{E}_j \rangle.
\end{aligned} \quad (50)$$

Note the first term on the right hand side, i.e.  $\langle \tau_-^{k+1}(\mathcal{L}_m), \mathcal{E}_i \rangle$  can be computed recursively using equation (37).

### 3.4 Computing $\mathcal{L}_1 F_2$

We are now ready to compute  $\mathcal{L}_1 F_2$  where  $F_2$  is given by Theorem 3.1. First observe that for  $\mathcal{L}_1$ , Lemma 3.2 has a simpler form

**Lemma 3.3** *For all  $i$  and  $j$ ,*

$$\begin{aligned} (i) \quad & \mathcal{L}_1 u_i = -u_i^2, \\ (ii) \quad & \mathcal{L}_1 g_i = 6u_i g_i, \\ (iii) \quad & \mathcal{L}_1 r_{ij} = \frac{15}{4} \delta_{ij} + (u_i + u_j) r_{ij} - \sum_k v_{ik} v_{jk}. \end{aligned}$$

Using this lemma and the definition of  $v_{ij}$ ,  $\theta_{ij}$  and  $\Omega_{ij}$ , we obtain for  $i \neq j$ ,

$$\mathcal{L}_1 v_{ij} = (u_i - u_j) \sum_k v_{ik} v_{jk}, \quad (51)$$

$$\mathcal{L}_1 \theta_{ij} = (3u_i + u_j) \theta_{ij} - \frac{11}{4} r_{ij} + \sum_{k,l} r_{ik} v_{jl} v_{kl}, \quad (52)$$

and

$$\mathcal{L}_1 \Omega_{ij} = 3(u_i + u_j) \Omega_{ij} - \frac{11}{4} \sum_k r_{ik} r_{jk} + \sum_{k,l,p} r_{il} r_{jk} v_{kp} v_{lp}. \quad (53)$$

In particular

$$\begin{aligned} & \mathcal{L}_1 \left\{ \Omega_{ij} \left( \frac{1}{\sqrt{g_i g_j}} - \frac{1}{g_j} \sqrt{\frac{g_i}{g_j}} \right) \right\} \\ = & 6 \left( \theta_{ij} - \theta_{ji} + \sum_{k,l} r_{il} r_{jk} v_{kl} \right) \frac{1}{g_j} \sqrt{\frac{g_i}{g_j}} \\ & + \left( -\frac{11}{4} \sum_k r_{ik} r_{jk} + \sum_{k,l,p} r_{il} r_{jk} v_{kp} v_{lp} \right) \left( \frac{1}{\sqrt{g_i g_j}} - \frac{1}{g_j} \sqrt{\frac{g_i}{g_j}} \right). \end{aligned} \quad (54)$$

Therefore  $\mathcal{L}_1 F_2$  does not contain second order poles. Moreover by equation (37),

$$\begin{aligned} \langle \tau_-^m(\mathcal{L}_1), \mathcal{E}_i \rangle &= -u_i^2 \langle \tau_-^m(\mathcal{S}), \mathcal{E}_i \rangle - (2m+1) u_i \langle \tau_-^{m-1}(\mathcal{S}), \mathcal{E}_i \rangle \\ &\quad - \sum_j (u_j^2 - u_i^2) r_{ij} \sqrt{\frac{g_i}{g_j}} \langle \tau_-^{m-1}(\mathcal{S}), \mathcal{E}_j \rangle \\ &\quad - (m^2 - \frac{1}{4}) \langle \tau_-^{m-2}(\mathcal{S}), \mathcal{E}_i \rangle - 2m \sum_j v_{ij} \sqrt{\frac{g_i}{g_j}} \langle \tau_-^{m-2}(\mathcal{S}), \mathcal{E}_j \rangle \\ &\quad - \sum_{j,k} v_{ij} v_{jk} \sqrt{\frac{g_i}{g_k}} \langle \tau_-^{m-2}(\mathcal{S}), \mathcal{E}_k \rangle \end{aligned} \quad (55)$$

for  $m \geq 1$ . So by equation (50),

$$\begin{aligned}
\mathcal{L}_1 \langle \tau_-^m(\mathcal{S}), \mathcal{E}_i \rangle &= 2(m+3)u_i \langle \tau_-^m(\mathcal{S}), \mathcal{E}_i \rangle + \left\{ (m+1)^2 - \frac{1}{4} \right\} \langle \tau_-^{m-1}(\mathcal{S}), \mathcal{E}_i \rangle \\
&\quad + 2(m+1) \sum_j v_{ij} \sqrt{\frac{g_i}{g_j}} \langle \tau_-^{m-1}(\mathcal{S}), \mathcal{E}_j \rangle \\
&\quad + \sum_{j,k} v_{ij} v_{jk} \sqrt{\frac{g_i}{g_k}} \langle \tau_-^{m-1}(\mathcal{S}), \mathcal{E}_k \rangle
\end{aligned} \tag{56}$$

for  $m \geq 1$ . In particular,

$$\begin{aligned}
\mathcal{L}_1 \langle \tau_-^2(\mathcal{S}), \mathcal{E}_i \rangle &= 10u_i \langle \tau_-^2(\mathcal{S}), \mathcal{E}_i \rangle + \frac{35}{4} \sum_j r_{ij} \sqrt{g_i g_j} + 6 \sum_{j,k} v_{ij} r_{jk} \sqrt{g_i g_k} \\
&\quad + \sum_{j,k,l} v_{ij} v_{jk} r_{kl} \sqrt{g_i g_l}
\end{aligned} \tag{57}$$

and

$$\begin{aligned}
\mathcal{L}_1 \left( \frac{1}{g_i^2} \langle \tau_-^3(\mathcal{S}), \mathcal{E}_i \rangle \right) &= \frac{63}{4} \frac{\langle \tau_-^2(\mathcal{S}), \mathcal{E}_i \rangle}{g_i^2} + 8 \sum_j v_{ij} \frac{1}{\sqrt{g_i^3 g_j}} \langle \tau_-^2(\mathcal{S}), \mathcal{E}_j \rangle \\
&\quad + \sum_{j,k} v_{ij} v_{jk} \frac{1}{\sqrt{g_i^3 g_k}} \langle \tau_-^2(\mathcal{S}), \mathcal{E}_k \rangle.
\end{aligned} \tag{58}$$

The last equation implies that  $\mathcal{L}_1 F_2$  does not contain  $\langle \tau_-^3(\mathcal{S}), \mathcal{E}_i \rangle$ .

Using Lemma 3.3 and equations (51), (52), (54), (57), (58), a straightforward computation shows that

$$\begin{aligned}
1152 \mathcal{L}_1 F_2 &= 1152 \langle \mathcal{L}_1 \rangle_2 \\
&= \sum_i \frac{\langle \tau_-^2(\mathcal{S}), \mathcal{E}_i \rangle}{g_i^2} \left( 6 + 24 \sum_j v_{ij}^2 \right) \\
&\quad + \sum_i \sum_{j \neq i} \theta_{ij} \left\{ 6 \frac{1}{g_j} \sqrt{\frac{g_i}{g_j}} + 48 \frac{1}{g_i} \sum_k v_{ik} v_{jk} + 24 \frac{1}{\sqrt{g_i g_j}} \sum_k v_{ik}^2 + 24 \frac{1}{g_j} \sqrt{\frac{g_i}{g_j}} \sum_k v_{jk}^2 \right\} \\
&\quad - \sum_i 72 r_{ii}^2 \frac{1}{g_i} + \sum_{i,j} \left\{ 57 r_{ij}^2 \frac{1}{g_i} - 144 r_{ii} r_{ij} v_{ij} \frac{1}{g_i} \right\} \\
&\quad + \sum_{i,j,k} \left\{ \frac{11}{4} r_{ij} r_{ik} \frac{1}{\sqrt{g_j g_k}} + 66 r_{ij} r_{ik} v_{ik} \frac{1}{\sqrt{g_i g_j}} - 36 (r_{ij} v_{ij}) (r_{ik} v_{ik}) \frac{1}{g_i} \right. \\
&\quad \quad \left. - 288 r_{jk}^2 v_{ij} v_{ik} \frac{1}{\sqrt{g_j g_k}} + 240 r_{jk}^2 v_{ij}^2 \frac{1}{g_j} \right\} \\
&\quad + \sum_{i,j,k,l} \left\{ -24 r_{jl} r_{kl} v_{ij} v_{ik} \frac{1}{g_l} + 24 r_{jk} r_{kl} v_{ij}^2 \frac{1}{\sqrt{g_j g_l}} \right\}
\end{aligned}$$

$$\begin{aligned}
& -576r_{jk}(r_{kl}v_{kl})v_{ij}v_{ik}\frac{1}{g_k} + 288r_{jk}(r_{kl}v_{kl})v_{ij}^2\frac{1}{\sqrt{g_jg_k}} \Big\} \\
& + \sum_{i,j,k,l,p} \left\{ -r_{jp}r_{kl}v_{ij}v_{ik}\frac{1}{\sqrt{g_lg_p}} - 24r_{jp}(r_{kl}v_{kl})v_{ij}v_{ik}\frac{1}{\sqrt{g_kg_p}} \right. \\
& \left. - 144(r_{jp}v_{jp})(r_{kl}v_{kl})v_{ij}v_{ik}\frac{1}{\sqrt{g_jg_k}} \right\}. \tag{59}
\end{aligned}$$

On the other hand, plugging the formulas (41), (43) and (44) into equation (32), then using the fact that

$$\sum_i \sum_{j \neq i} 6 \frac{1}{\sqrt{g_i g_j}} \theta_{ij} = \sum_i \sum_{j \neq i} 3 \frac{1}{\sqrt{g_i g_j}} (\theta_{ij} + \theta_{ji}) = - \sum_i \sum_{j \neq i} 3 \frac{1}{\sqrt{g_i g_j}} \sum_k r_{ik} r_{jk}$$

to simplify the resulting expression, we see that the prediction for the genus-2  $L_1$ -constraint is precisely given by equation (59). This completes the proof of Theorem 0.2.

## A Appendix: Another proof to Theorem 0.2

In this appendix, we describe an alternative proof to Theorem 0.2 using the formula for  $\langle\langle \mathcal{L}_1 \rangle\rangle_2$  given by Theorem 2.4. The advantage of this approach is that we do not need to know the precise formula for expressing  $F_2$  in terms of rotation coefficients. Therefore this approach is closer to the treatment of the genus-1 Virasoro conjecture described in [L4]. Moreover, in this approach, we are mainly dealing with vector field  $T(\overline{\mathcal{X}})$  which behaves much better than  $\mathcal{L}_1$ . Properties for  $T(\overline{\mathcal{X}})$  derived here may also be useful for the study of higher genus Gromov-Witten invariants.

Recall that by Theorem 2.4,

$$\begin{aligned}
\langle\langle \mathcal{L}_1 \rangle\rangle_2 &= \frac{1}{2} T(\overline{\mathcal{X}}) \left\{ \sum_{i=1}^N u_i B(\mathcal{E}_i, \mathcal{E}_i, \mathcal{E}_i) \right\} - \frac{1}{2} \sum_{i=1}^N u_i^2 B(\mathcal{E}_i, \mathcal{E}_i, \mathcal{E}_i) \\
&+ A_1(\tau_-^2(\mathcal{L}_1)) - T(\overline{\mathcal{X}}) \left\{ A_1 \left( \tau_-^2(\mathcal{L}_0) + \frac{3}{2} \tau_-(\mathcal{S}) \right) \right\}. \tag{60}
\end{aligned}$$

In this section, we will prove that the right hand side of equation (60) is equal to the right hand side of equation (59) up to a multiplicative constant 1152. As observed at the end of last section, this proves Theorem 0.2.

To get rid of  $T(\overline{\mathcal{X}})$  in the expression of  $\langle\langle \mathcal{L}_1 \rangle\rangle_2$  in equation (60), we need the following properties of this vector field:

$$\begin{aligned}
T(\overline{\mathcal{X}}) z_{i_1 \dots i_{k+2}} &= \sum_{m=2}^{k-1} \sum_{1 \leq j_1 < \dots < j_m \leq k} \sum_{p,q} \frac{u_p}{g_q} z_{p i_{j_1} \dots i_{j_m} q} z_{q i_1 \dots \widehat{i_{j_1}} \dots \widehat{i_{j_2}} \dots \dots \widehat{i_{j_m}} \dots i_{k+2}} \\
&+ \delta_{i_{k+1} i_{k+2}} \sum_p u_p z_{p i_1 \dots i_{k+1}} - (u_{i_{k+1}} + u_{i_{k+2}}) z_{i_1 \dots i_{k+2}} \tag{61}
\end{aligned}$$

and

$$\begin{aligned}
T(\overline{\mathcal{X}})\phi_{i_1 \dots i_k} &= \sum_{m=2}^k \sum_{1 \leq j_1 < \dots < j_m \leq k} \sum_{p,q} \frac{u_p}{g_q} z_{p i_{j_1} \dots i_{j_m} q} \phi_{q i_1 \dots \widehat{i_{j_1}} \dots \widehat{i_{j_2}} \dots \dots \widehat{i_{j_m}} \dots i_k} \\
&+ \frac{1}{24} \sum_{p,q} \frac{u_p}{g_q} z_{p i_1 \dots i_k q q}.
\end{aligned} \tag{62}$$

These two equations are obtained from equation (28) and (29) since

$$\begin{aligned}
T(\overline{\mathcal{X}}) \langle\langle \mathcal{W}_1, \dots \mathcal{W}_k \rangle\rangle_g &= \langle\langle T(\overline{\mathcal{X}}) \mathcal{W}_1 \dots \mathcal{W}_k \rangle\rangle_g \\
&+ \sum_{j=1}^k \langle\langle \mathcal{W}_1 \dots (\nabla_{T(\overline{\mathcal{X}})} \mathcal{E}_i) \dots \mathcal{W}_k \rangle\rangle_g,
\end{aligned}$$

and  $\nabla_{T(\overline{\mathcal{X}})} \mathcal{E}_i = -\overline{\mathcal{X}} \circ \mathcal{E}_i = -u_i \mathcal{E}_i$ . Moreover

$$T(\overline{\mathcal{X}}) g_i = -2 \langle \overline{\mathcal{X}}, \mathcal{E}_i \rangle = -2u_i g_i$$

for any  $i$ .

Using the above properties of  $T(\overline{\mathcal{X}})$ , we can get rid of this vector field in the right hand side of equation (60). We separate the contributions from tensor  $A_1$  and tensor  $B$ . Write

$$\langle\langle \mathcal{L}_1 \rangle\rangle_2 = L_A + L_B$$

where

$$L_A := A_1(\tau_-^2(\mathcal{L}_1)) - T(\overline{\mathcal{X}}) \left\{ A_1 \left( \tau_-^2(\mathcal{L}_0) + \frac{3}{2} \tau_-(\mathcal{S}) \right) \right\} \tag{63}$$

is the contribution from the tensor  $A_1$ , and

$$L_B := \frac{1}{2} \sum_i \left( u_i T(\overline{\mathcal{X}}) B(\mathcal{E}_i, \mathcal{E}_i, \mathcal{E}_i) - u_i^2 B(\mathcal{E}_i, \mathcal{E}_i, \mathcal{E}_i) \right) \tag{64}$$

is the contribution from the tensor  $B$ . We have

$$\begin{aligned}
2L_B &= \sum_{i,j,k} \frac{u_i u_j}{g_j g_k} \frac{1}{240} z_{ijjjjjkk} + \sum_{i,j,k,p} \frac{u_i u_p}{g_j g_k} \left( -\frac{1}{480} z_{iijjkkp} - \frac{1}{2880} z_{iiijkkp} \right) \\
&+ \sum_{i,j,k,p,q} \frac{u_i u_p}{g_j g_k g_q} \left( \frac{1}{2880} z_{iiijjk} z_{kpqq} - \frac{1}{480} z_{iiij} z_{jkppqq} + \frac{1}{320} z_{ijjk} z_{iikppq} \right. \\
&\quad \left. - \frac{1}{80} z_{iiijk} z_{ijkppq} + \frac{1}{240} z_{iiijk} z_{jkppqq} - \frac{1}{320} z_{iiijk} z_{ikppqq} \right) \\
&+ \sum_i \phi_i \left\{ \sum_{j,k} \left( \frac{1}{10} \frac{u_i u_j}{g_i g_k} z_{iiijkk} + \frac{1}{10} \frac{u_j u_k}{g_i g_k} z_{ijkkkk} + \frac{1}{120} \frac{u_k(2u_i + 2u_j - 3u_k)}{g_i g_j} z_{ijjjkk} \right) \right. \\
&\quad \left. - \sum_{j,k,p} \frac{u_k u_p}{g_i g_j} \left( \frac{11}{120} z_{ijjjkkp} + \frac{1}{120} z_{ijkkkp} \right) \right\}
\end{aligned}$$

$$\begin{aligned}
& + \sum_{j,k,p,q} \frac{u_p u_q}{g_i g_j g_k} \left( \frac{1}{40} z_{ijqq} z_{jkkp} - \frac{11}{120} z_{ijkkp} z_{jqqq} - \frac{1}{24} z_{ikqq} z_{jjkpq} \right. \\
& \quad \left. + \frac{7}{60} z_{ijkp} z_{jkqq} + \frac{1}{60} z_{ijkq} z_{jkpq} - \frac{1}{15} z_{ikpq} z_{jjkq} \right. \\
& \quad \left. - \frac{17}{60} z_{ijkpq} z_{jkq} + \frac{3}{40} z_{ikpq} z_{jjkq} \right) \Big\} \\
& - \sum_i \phi_{ii} \sum_{j,k} \frac{3}{10} \frac{u_i u_j}{g_i g_k} z_{iijk} \\
& + \sum_{i,j} \phi_{ij} \left\{ \sum_k \left( \frac{2}{5} \frac{u_i u_k}{g_i g_j} z_{iiijk} + \frac{1}{10} \frac{u_k(2u_i + 2u_j - 3u_k)}{g_i g_j} z_{ijkkk} \right. \right. \\
& \quad \left. \left. - \frac{3}{40} \frac{u_i(2u_j + 2u_k - 3u_i)}{g_j g_k} z_{iijk} \right) \right. \\
& \quad - \sum_{k,p} \left( \frac{1}{40} \frac{u_i u_p}{g_j g_k} (z_{ijkkp} + z_{iijkp}) + \frac{1}{120} \frac{u_k u_p}{g_i g_j} z_{ikkkp} \right) \\
& \quad + \sum_{k,p,q} \left( \frac{1}{40} \frac{u_p u_q}{g_i g_j g_k} (-4z_{ikkp} z_{jqqq} - 16z_{ikpq} z_{jkqq} + 3z_{ipqq} z_{jkkq}) \right. \\
& \quad \left. \left. + \frac{1}{40} \frac{u_i u_p}{g_j g_k g_q} (-5z_{iijk} z_{kpqq} - 18z_{iikq} z_{jkpq} + 6z_{ikq} z_{ijkp}) \right) \right\} \\
& + \sum_i \phi_{iii} \sum_{j,k} \frac{1}{10} \frac{u_i u_j}{g_i g_k} z_{ijkk} \\
& + \sum_{i,j} \phi_{ij} \left\{ \sum_k \left( \frac{3}{5} \frac{u_i u_k}{g_i g_j} z_{iijk} - \frac{1}{20} \frac{u_k(2u_i + 2u_j - 3u_k)}{g_i g_j} z_{jkkk} \right. \right. \\
& \quad \left. \left. + \frac{3}{40} \frac{u_i(2u_j - u_i)}{g_j g_k} z_{ijkk} \right) \right. \\
& \quad \left. + \sum_{k,p} \left( -\frac{1}{10} \left( \frac{u_i u_p}{g_j g_k} + \frac{u_k u_p}{g_i g_j} \right) z_{jkkp} + \frac{1}{20} \frac{u_i u_p}{g_j g_k} z_{ijkp} \right) \right\} \\
& + \sum_{i,j,k} \phi_{ijk} \left\{ -\frac{3}{10} \frac{u_i(2u_j + 2u_k - 3u_i)}{g_j g_k} z_{iijk} - \sum_p \frac{u_i u_p}{g_j g_k} \left( \frac{1}{40} z_{iikp} + \frac{1}{2} z_{ijkp} \right) \right\} \\
& + \sum_i \phi_{iiii} \cdot \frac{1}{10} \frac{u_i^2}{g_i} - \sum_{i,j} \left( \frac{1}{120} \phi_{iiij} + \frac{1}{20} \phi_{iijj} \right) \frac{u_i(2u_j - u_i)}{g_j} \\
& + \sum_{i,j} \phi_i \phi_j \left\{ \sum_k \left( \frac{12}{5} \frac{u_i u_k}{g_i g_j} z_{iiijk} + \frac{1}{5} \frac{u_k(2u_i + 2u_j - 3u_k)}{g_i g_j} z_{ijkkk} \right) \right. \\
& \quad \left. - \sum_{k,p} \frac{u_k u_p}{g_i g_j} z_{ijkkp} - \sum_{k,p,q} \frac{u_p u_q}{g_i g_j g_k} \left( \frac{4}{5} z_{ikpq} z_{jkqq} + z_{ijkp} z_{kqq} \right) \right\} \\
& - \sum_{i,j} \phi_i \phi_{jj} \sum_k \frac{36}{5} \frac{u_j u_k}{g_i g_j} z_{ijjk}
\end{aligned}$$

$$\begin{aligned}
& + \sum_{i,j} \phi_i \phi_{ij} \left\{ \sum_k \left( \frac{36}{5} \frac{u_i u_k}{g_i g_j} z_{iijk} - \frac{6}{5} \frac{u_k(2u_i + 2u_j - 3u_k)}{g_i g_j} z_{jkkk} \right) \right. \\
& \quad \left. - \sum_{k,p} \frac{12}{5} \frac{u_k u_p}{g_i g_j} z_{jkkp} \right\} \\
& - \sum_{i,j,k} \phi_i \phi_{jk} \cdot \frac{6}{5} \frac{u_j(2u_i + 2u_k - 3u_j)}{g_i g_k} z_{ijjk} \\
& + \sum_i \phi_i \phi_{iii} \cdot \frac{12}{5} \frac{u_i^2}{g_i} - \sum_{i,j} \phi_i \phi_{ijj} \cdot \frac{6}{5} \frac{u_j(2u_i - u_j)}{g_i} \\
& - \sum_i \phi_{ii}^2 \cdot \frac{18}{5} \frac{u_i^2}{g_i} + \sum_{i,j} \phi_{ij}^2 \cdot \frac{9}{5} \frac{u_i(2u_j - u_i)}{g_j}.
\end{aligned}$$

To compute the contribution from tensor  $A_1$ , we also need to understand the action of  $T(\overline{\mathcal{X}})$  on coefficients of  $\tau_-^k(\mathcal{L}_0)$  and  $\tau_-^k(\mathcal{S})$  along idempotents. Since

$$\nabla_{T(\mathcal{W})} \tau_-^k(\mathcal{S}) = \tau_-^k(\nabla_{T(\mathcal{W})} \mathcal{S}) = -\tau_-^{k+1}(T(\mathcal{W})) = -\tau_-^k(\mathcal{W}),$$

by equation (48),

$$T(\mathcal{W}) \langle \tau_-^k(\mathcal{S}), \mathcal{E}_i \rangle = - \langle \tau_-^k(\mathcal{W}), \mathcal{E}_i \rangle - \langle \tau_-^k(\mathcal{S}), \mathcal{W} \circ \mathcal{E}_i \rangle$$

for any vector field  $\mathcal{W}$  and  $k \geq 0$ . In particular, for  $k \geq 1$ ,

$$T(\overline{\mathcal{X}}) \langle \tau_-^k(\mathcal{S}), \mathcal{E}_i \rangle = -u_i \langle \tau_-^k(\mathcal{S}), \mathcal{E}_i \rangle. \quad (65)$$

Comparing to equation (56), we see that  $T(\overline{\mathcal{X}})$  behaves much better than  $\mathcal{L}_1$ . Similarly, since  $\nabla_{T(\mathcal{W})} \mathcal{L}_0 = R(\mathcal{W})$  (cf. [L2, Equation (43)]),

$$T(\mathcal{W}) \langle \tau_-^k(\mathcal{L}_0), \mathcal{E}_i \rangle = \langle \tau_-^k R(\mathcal{W}), \mathcal{E}_i \rangle - \langle \tau_-^k(\mathcal{L}_0), \mathcal{W} \circ \mathcal{E}_i \rangle$$

for any vector field  $\mathcal{W}$ . By [L2, Lemma 3.11], for primary vector field  $\overline{\mathcal{W}}$  and  $k \geq 2$ ,

$$\tau_-^k R(\overline{\mathcal{W}}) = \tau_-^{k-1} (R\tau_-(\overline{\mathcal{W}}) + \mathcal{G} * \overline{\mathcal{W}} + \overline{\mathcal{W}}) = 0.$$

So for  $k \geq 2$ ,

$$T(\overline{\mathcal{X}}) \langle \tau_-^k(\mathcal{L}_0), \mathcal{E}_i \rangle = -u_i \langle \tau_-^k(\mathcal{L}_0), \mathcal{E}_i \rangle. \quad (66)$$

Therefore for

$$\mathcal{W} = \tau_-^k(\mathcal{S}) \text{ or } \tau_-^{k+1}(\mathcal{L}_0) \text{ with } k \geq 1,$$

we can use equation (65) and equation (66) to compute  $T(\overline{\mathcal{X}})A_1(\mathcal{W})$  and obtain

$$\begin{aligned}
T(\overline{\mathcal{X}})A_1(\mathcal{W}) &= \sum_i \frac{u_i \langle \mathcal{W}, \mathcal{E}_i \rangle}{g_i^2} \left( \frac{21}{10} \phi_i^2 + \frac{3}{10} \phi_{ii} \right) - \sum_{i,j} \frac{1}{240} \frac{\langle \mathcal{W}, \mathcal{E}_i \rangle}{g_i} \frac{u_i + 2u_j}{g_j} \phi_{ij} \\
&+ \sum_i \phi_i \left\{ \frac{3}{20} \frac{u_i \langle \tau_-(\mathcal{W}), \mathcal{E}_i \rangle}{g_i^2} + \sum_{j,k,p} \frac{1}{20} \frac{\langle \mathcal{W}, \mathcal{E}_k \rangle}{g_k} \frac{u_p}{g_i g_j} z_{ijkp} \right\}
\end{aligned}$$

$$\begin{aligned}
& + \sum_{j,k} \left( \left( \frac{7}{120} \frac{\langle \mathcal{W}, \mathcal{E}_i \rangle}{g_i} + \frac{1}{10} \frac{\langle \mathcal{W}, \mathcal{E}_k \rangle}{g_k} \right) \frac{u_j}{g_i g_k} z_{ijkk} \right. \\
& \quad \left. + \frac{13}{240} \frac{\langle \mathcal{W}, \mathcal{E}_k \rangle}{g_k} \frac{2u_i + u_k}{g_i g_j} z_{ijjk} \right) \Bigg\} \\
& + \sum_i \frac{\langle \mathcal{W}, \mathcal{E}_i \rangle}{g_i} \left\{ \sum_{j,k} \left( \frac{1}{240} \frac{u_j}{g_i g_k} z_{iijk} + \frac{1}{960} \frac{u_i + 2u_j}{g_j g_k} z_{ijjkk} \right) \right. \\
& \quad + \sum_{j,k,p} \frac{1}{1152} \frac{u_p}{g_j g_k} z_{ijkkp} \\
& \quad \left. + \sum_{j,k,p,q} \frac{1}{480} \frac{u_p}{g_j g_k g_q} \left( \frac{19}{12} z_{ijjk} z_{kpqq} + z_{ijpq} z_{jkkq} \right) \right\} \\
& + \sum_i \frac{\langle \tau_-(\mathcal{W}), \mathcal{E}_i \rangle}{g_i} \left\{ \sum_j \frac{1}{160} \frac{u_i}{g_i g_j} z_{iijj} + \sum_{j,k,p} \frac{1}{1152} \frac{u_p}{g_j g_k} z_{ijkp} \right. \\
& \quad \left. + \sum_{j,k} \left( \frac{1}{480} \frac{u_j}{g_i g_k} z_{ijkk} + \frac{1}{1152} \frac{u_i + 2u_j}{g_j g_k} z_{ijkk} + \frac{1}{480} \frac{u_k}{g_i g_j} z_{iijk} \right) \right\} \\
& + \sum_i \frac{\langle \tau_-^2(\mathcal{W}), \mathcal{E}_i \rangle}{g_i} \cdot \frac{1}{384} \frac{u_i}{g_i}.
\end{aligned}$$

Define

$$c_{ij;k} := \frac{1}{g_i} \left\{ \langle \tau_-^k(\mathcal{L}_1), \mathcal{E}_i \rangle - (u_i + 2u_j) \left\langle \tau_-^k(\mathcal{L}_0) + \frac{3}{2} \tau_-^{k-1}(\mathcal{S}), \mathcal{E}_i \right\rangle \right\} \quad (67)$$

and

$$d_{i;k} := \frac{1}{g_i} \left\langle \tau_-^k(\mathcal{L}_0) + \frac{3}{2} \tau_-^{k-1}(\mathcal{S}), \mathcal{E}_i \right\rangle. \quad (68)$$

By equations (38) and (55),

$$\begin{aligned}
c_{ij;k} &= 2u_i u_j \frac{\langle \tau_-^k(\mathcal{S}), \mathcal{E}_i \rangle}{g_i} - \{(k+2)u_i + 2(1-k)u_j\} \frac{\langle \tau_-^{k-1}(\mathcal{S}), \mathcal{E}_i \rangle}{g_i} \\
& + \sum_p (u_p - 2u_j)(u_i - u_p) r_{ip} \frac{1}{\sqrt{g_i g_p}} \langle \tau_-^{k-1}(\mathcal{S}), \mathcal{E}_p \rangle \\
& + \left( \frac{1}{4} - k^2 \right) \frac{\langle \tau_-^{k-2}(\mathcal{S}), \mathcal{E}_i \rangle}{g_i} + 2k \sum_p (u_i - u_p) r_{ip} \frac{1}{\sqrt{g_i g_p}} \langle \tau_-^{k-2}(\mathcal{S}), \mathcal{E}_p \rangle \\
& - \sum_{p,q} (u_i - u_p)(u_p - u_q) r_{ip} r_{pq} \frac{1}{\sqrt{g_i g_q}} \langle \tau_-^{k-2}(\mathcal{S}), \mathcal{E}_q \rangle
\end{aligned}$$

and

$$d_{i;k} = -u_i \frac{\langle \tau_-^k(\mathcal{S}), \mathcal{E}_i \rangle}{g_i} + (1-k) \frac{\langle \tau_-^{k-1}(\mathcal{S}), \mathcal{E}_i \rangle}{g_i} - \sum_p (u_p - u_i) r_{ip} \frac{1}{\sqrt{g_i g_p}} \langle \tau_-^{k-1}(\mathcal{S}), \mathcal{E}_p \rangle.$$

The contribution to  $\langle\langle \mathcal{L}_1 \rangle\rangle_2$  from tensor  $A_1$  is

$$\begin{aligned}
L_A = & \sum_i \frac{7}{10} \frac{1}{g_i} c_{ii;2} \phi_i^2 + \sum_i \frac{1}{10} \frac{1}{g_i} c_{ii;2} \phi_{ii} - \sum_{i,j} \frac{1}{240} \frac{1}{g_j} c_{ij;2} \phi_{ij} \\
& + \sum_i \phi_i \left\{ \frac{1}{20} \frac{1}{g_i} c_{ii;3} + \sum_{j,k} \left( \frac{13}{240} \frac{1}{g_i g_j} c_{ki;2} z_{ijjk} - \frac{7}{120} \frac{u_j}{g_i g_k} d_{i;2} z_{ijkk} - \frac{1}{10} \frac{u_j}{g_i g_k} d_{k;2} z_{ijkk} \right) \right. \\
& \quad \left. - \sum_{j,k,p} \frac{1}{20} \frac{u_p}{g_i g_j} d_{k;2} z_{ijkp} \right\} \\
& + \sum_i \frac{1}{1152} \frac{1}{g_i} c_{ii;4} + \sum_{i,j} \frac{1}{480} \frac{1}{g_i g_j} c_{ii;3} z_{iijj} \\
& + \sum_{i,j,k} \frac{1}{1152} \frac{1}{g_j g_k} c_{ij;3} z_{ijkk} + \sum_{i,j,k} \frac{1}{960} \frac{1}{g_j g_k} c_{ij;2} z_{ijjkk} \\
& - \sum_i d_{i;3} \left\{ \sum_{j,k} \frac{1}{480} \left( \frac{u_j}{g_i g_k} z_{ijkk} + \frac{u_k}{g_i g_j} z_{iijk} \right) + \sum_{j,k,p} \frac{1}{1152} \frac{u_p}{g_j g_k} z_{ijkp} \right\} \\
& - \sum_i d_{i;2} \left\{ \sum_{j,k} \frac{1}{240} \frac{u_j}{g_i g_k} z_{iijkk} + \sum_{j,k,p} \frac{1}{1152} \frac{u_p}{g_j g_k} z_{ijkkp} \right. \\
& \quad \left. + \sum_{j,k,p,q} \frac{1}{480} \frac{u_p}{g_j g_k g_q} \left( \frac{19}{12} z_{ijjk} z_{kpqq} + z_{ijpq} z_{jkkq} \right) \right\}.
\end{aligned}$$

In section 3.2, we have described how to represent genus-0 and genus-1 functions  $z_{i_1 \dots i_k}$  and  $\phi_{i_1 \dots i_k}$  in terms of rotation coefficients. For our purpose, we only need  $z_{i_1 \dots i_k}$  for  $4 \leq k \leq 7$  and  $\phi_{i_1 \dots i_k}$  for  $1 \leq k \leq 4$ . Using these formulas, we can express  $L_A + L_B$  in terms of functions  $u_i$ ,  $g_i$ ,  $r_{ij}$ , and  $\langle \tau_-^k(\mathcal{S}), \mathcal{E}_i \rangle$  with  $k \geq 2$ . After lengthy but straightforward computations, we can check that  $L_A + L_B$  is equal to the right hand side of equation (59) up to a multiplicative constant 1152, and therefore Theorem 0.2 is proved. It is interesting to observe what happens to terms  $\langle \tau_-^k(\mathcal{S}), \mathcal{E}_i \rangle$  in the computation. Both  $L_A$  and  $L_B$  contains terms  $\langle \tau_-^k(\mathcal{S}), \mathcal{E}_i \rangle$  with  $2 \leq k \leq 4$ . But coefficients of  $\langle \tau_-^3(\mathcal{S}), \mathcal{E}_i \rangle$  (as well as  $\langle \tau_-^4(\mathcal{S}), \mathcal{E}_i \rangle$ ) from  $L_A$  and  $L_B$  are opposite to each other, therefore are cancelled in  $L_A + L_B$ . However  $L_A + L_B$  contains  $\langle \tau_-^2(\mathcal{S}), \mathcal{E}_i \rangle$ , which exactly match the corresponding terms in the right hand side of equation (59). In a contrast, the formula for  $F_2$  in Theorem 3.1 contains  $\langle \tau_-^3(\mathcal{S}), \mathcal{E}_i \rangle$ . The action of  $\mathcal{L}_1$  transforms this term to expressions only involve  $\langle \tau_-^2(\mathcal{S}), \mathcal{E}_i \rangle$  as indicated in equation (58).

## References

- [BP] Belorousski, P. and Pandharipande, R., *A descendent relation in genus 2*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 29 (2000) 171-191.
- [CK] Cox, D. and Katz, S., *Mirror symmetry and algebraic geometry*, Providence, R.I. AMS, 1999.

- [D] B. Dubrovin, *Geometry of 2D topological field theories*, Integrable Systems and Quantum Groups, Springer Lectures Notes in Math. 1620 (1996), 120-348.
- [DZ1] Dubrovin, B., Zhang, Y., *Bihamiltonian hierarchies in 2D topological field theory at one-loop approximation*, Comm. Math. Phys. 198 (1998), no.2, 311-361.
- [DZ2] Dubrovin, B., Zhang, Y., *Frobenius manifolds and Virasoro constraints*, Selecta Math. (N.S.) 5 (1999) 423-466.
- [DZ3] Dubrovin, B., Zhang, Y., *Normal forms of hierarchies of integrable PDEs, Frobenius manifolds and Gromov-Witten invariants*, math.DG/0108160.
- [EHX] Eguchi, T., Hori, K., and Xiong, C., *Quantum Cohomology and Virasoro Algebra*, Phys. Lett. B402 (1997) 71-80.
- [Ge1] Getzler, E., *Topological recursion relations in genus 2*, Integrable systems and algebraic geometry (Kobe/kyoto, 1997) 73-106.
- [Ge2] Getzler, E., *The Virasoro conjecture for Gromov-Witten invariants*, (math.AG/9812026)
- [Gi1] Givental, A., *Semisimple Frobenius structures at higher genus*, Intern. Math. Res. Notices, 23 (2001), 1265-1286.
- [Gi2] Givental, A., *Gromov-Witten invariants and quantization of quadratic hamiltonians*, Moscow Mathematical Journal, v.1, no. 4 (2001), 551-568.
- [Gi3] Givental, A., *Symplectic geometry of Frobenius structures*, math.AG/0305409.
- [Ko] Kontsevich, M., *Intersection theory on the moduli space of curves and the matrix airy function*, Comm. Math. Phys., 147, 1-23 (1992).
- [LiT] Li, J. and Tian, G., *Virtual moduli cycles and Gromov-Witten invariants of general symplectic manifolds*, Topics in symplectic 4-manifolds (Irvine, CA, 1996), 47-83.
- [L1] X. Liu, *Elliptic Gromov-Witten invariants and Virasoro conjecture*, Comm. Math. Phys. 216 (2001), 705-728.
- [L2] X. Liu, *Quantum product on the big phase space and Virasoro conjecture*, Advances in Mathematics 169 (2002), 313-375.
- [L3] Liu, X., *Quantum product, topological recursion relations, and the Virasoro conjecture*, to appear in Preceedings of Mathematical Society of Japan - 9th International Research Institute on "Integrable Systems in Differential Geometry" in 2000, Tokyo, Japan.
- [L4] X. Liu, *Idempotents on the big phase space*, math.DG/0310409.

- [LT] X. Liu and G. Tian, *Virasoro constraints for quantum cohomology*, J. Diff. Geom. 50 (1998), 537 - 591.
- [RT] Ruan, Y. and Tian, G., *Higher genus symplectic invariants and sigma models coupled with gravity*, Invent. Math. 130 (1997), 455-516.
- [W1] Witten, E., *Two dimensional gravity and intersection theory on Moduli space*, Surveys in Diff. Geom., 1 (1991), 243-310.
- [W2] Witten, E., *On the Kontsevich model and other models of two dimensional gravity*, in "Proceedings of the XXth international conference on differential geometric methods in theoretical physics (New York, 1991)", World Sci. Publishing, River Edge, NJ, 1992, pp. 176-216.

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